Index theory for manifolds with corners

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Sommaire

C^*-algebraic index

Let

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r] \arrow[d] & A_2 \arrow[r, "r"] \arrow[d] & A_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \arrow[r] \arrow[d] & B_2 \arrow[r, "\eta"] \arrow[d, "\sigma"] & B_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

B_1, B_2, B_3, C_1, C_2 and C_3 being unitary C^*-algebras. And $\delta : K_1(C_2) \to K_0(A_2)$ the index map of the second column.
From this diagram we can produce a diagonal exact sequence :

$$0 \longrightarrow A_1 \longrightarrow B_2 \xrightarrow{\sigma_{diag}=(\sigma,\eta)} C_2 \underset{C_3}{\oplus} B_3 \longrightarrow 0 \ .$$

Theorem : K-theoretical obstruction

Let $T \in B_2$ s.t $\sigma(T)$ invertible. Then the following are equivalents

- $\exists T' \in M_N(B_2)$ with $\sigma(T')$ invertible with $[\sigma(T)]_1 = [\sigma(T')]_1$ and $\eta(T')$ invertible
- $K_0(r)(\delta[\sigma(T)]_1) = 0.$

In particular $\sigma_{diag}(T')$ will be invertible in that case.

Boundary index : Application/Motivation

Application : Elliptic operators on a singular manifold ?

From a Lie groupoid $G \rightrightarrows G^{(0)}$ the pseudodifferential calculus defined on G fit in the exact sequence :

$$0 \longrightarrow C^*(G) \longrightarrow \overline{\psi^0}(G) \xrightarrow{\sigma} C_0(S^*G) \longrightarrow 0\ .$$

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0 \arrow[r] & C_0(S^*G_U) \arrow[r] \arrow[d] & C_0(S^*G) \arrow[r] \arrow[d] & C_0(S^*G_F) \arrow[r] \arrow[d] & 0 \\
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\end{tikzcd}$$

In this context, the boundary index is defined by :
$Ind_\partial := K_0(r) \circ \delta : K_1(C_0(S^*G)) \longrightarrow K_0(C^*(G_F))$.
The elliptic operators on G which are "diagonal invertible" will be called *Fully elliptic* And the

diagram chasing statement could be expressed as follows :

Theorem

$Ind_\partial([\sigma(T)]_1) = 0$ iff $\exists T' \in \overline{\psi^0}(G)$ Fully elliptic with $\sigma(T)$ and $\sigma(T')$ stably homotopic.

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$\rightarrow$ Need of a geometrical context and an adapted groupoid to arise singularities.

Geometrical framwork : manifold with corners

Definition : manifold with corners

A manifold with corners X is a topological space where each $x \in X$ has a neighborhood diffeomorphic to $\mathbb{R}^{k+} \times \mathbb{R}^{n-k}$. We denote $k := codim(x)$.

Connected components of $\{x \in X : codim(x) = k\}$ are called *codimension k faces*, their set is denoted $\mathscr{F}_k(X)$.

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Definition : manifold with embedded corners

Such a manifold is a topological space endowed with a sub algebra $:= C^\infty(X) \subset C^0(X)$ s.t :

- $\exists \widetilde{X}$ smooth manifold, $i : X \to \widetilde{X}$ with $i^* C^0(\widetilde{X}) = C^\infty(X)$
- $\exists (\rho_i)_{i=1}^N$, $\rho_i : \widetilde{X} \to \mathbb{R}$ smooth. s.t $i(X) = \bigcap\limits_{i \in [\![1,N]\!]} \{\rho_i \geqslant 0\}$
- $\forall J \subset [\![1,N]\!]$, $(d_x \rho_i)_{i \in J}$ are linearly independant for $x \in \bigcap\limits_{i \in J} \{\rho_i = 0\}$.

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Definition : family of manifolds with embedded corners

We call *Family of manifold with embedded corners* a submersion $\pi : \widetilde{X} \longrightarrow B$ from an embedded corner manifold to a smooth manifold such that : $\forall f \subset \widetilde{X}$ face, $\pi_{|f} : f \twoheadrightarrow B$ is still surjective.

For now I suppose also : $\mathcal{N}(\widetilde{X}, H_i) = H_i \times \mathbb{R}$.

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To a pair of embedded Lie groupoids
$$\begin{matrix} H & \subseteq & G \\ \downdownarrows & & \downdownarrows \\ H^{(0)} & \subseteq & G^{(0)} \end{matrix}$$
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Functoriality

If we restrict to Lie groupoid pair morphisms
$$\begin{array}{ccccccc} H_1 & \subseteq & G_1 & \xrightarrow{f} & G_2 & \supseteq & H_2 \\ \downdownarrows & & \downdownarrows & & \downdownarrows & & \downdownarrows \\ H_1^{(0)} & \subseteq & G_1^{(0)} & \xrightarrow{f^{(0)}} & G_2^{(0)} & \supseteq & H_2^{(0)} \end{array}$$
s.t

$(f^{(0)})^{-1}(H_2^{(0)}) = H_1^{(0)}$ and $f^{(0)} : (G_1^{(0)}, H_1^{(0)}) \to (G_2^{(0)}, H_2^{(0)})$ has its $d_{\mathcal{N}} f$ fiberwise injective.
Then the association is functorial.

Monthubert groupoid

We set :

$$\begin{cases} Mt_1(X) = & Bst(\widetilde{X} \underset{B}{\times} \widetilde{X}, H_1 \underset{B}{\times} H_1) \\ Mt_{k+1}(X) = & Bst\left(Mt_k(X),\, Bst(H_{k+1} \underset{B}{\times} H_{k+1}, \bigsqcup_{\alpha \leqslant k} H_{k+1,\alpha} \underset{B}{\times} H_{k+1,\alpha})\right). \end{cases}$$

We set $\Gamma_b(X) := Mt_N^c(X)_{|X}$ the Monthubert Puff groupoid.

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Computing it, we get :

$$\Gamma_b(X) = \mathring{X} \underset{B}{\times} \mathring{X} \underset{\sim}{\bigsqcup} \bigsqcup_{g \in \mathscr{F}_1} (g \underset{B}{\times} g \times \mathbb{R}^{*+}) \underset{\sim}{\bigsqcup} \bigsqcup_{f \in \mathscr{F}_2} (f \underset{B}{\times} f \times \mathbb{R}^{*+2}).$$

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Taking back our boundary index, our index map in this context corresponds to :

$$\begin{tikzcd} K_0(C^*(\Gamma_b(X))) \arrow[r, "K_0(r)"] & K_0\Big(C^*\big(\underbrace{\bigsqcup_{g \in \mathscr{F}_1} (g \underset{B}{\times} g \times \mathbb{R}^{*+}) \underset{\sim}{\bigsqcup} \bigsqcup_{f \in \mathscr{F}_2} (f \underset{B}{\times} f \times \mathbb{R}^{*+2})}_{:= \Gamma_b(X)_{|\mathscr{F}_1 \cup \mathscr{F}_2} = G_F}\big)\Big) \\ K_1(C_0(S^*G)) \arrow[u, bend left, "\delta"] & \end{tikzcd}$$

Computation of $K_*(C^*(\Gamma_b(X)_{|\mathscr{F}_1 \cup \mathscr{F}_2}))$

Remark :

$K_*(C^*(\Gamma_b(X)_{|\mathscr{F}_1})) = \bigoplus_{g \in \mathscr{F}_1} K_*(g \underset{B}{\times} g \times \mathbb{R}^{*+})) = K_{1-*}(C_0(B))^{\#\mathscr{F}_1}$, and the same way $K_*(C^*(\Gamma_b(X)_{|\mathscr{F}_2})) = K_*(C_0(B))^{\#\mathscr{F}_2}$.

From the exact sequence :

$$0 \longrightarrow C^*(\Gamma_b(X)_{|\mathscr{F}_1}) \longrightarrow C^*(\Gamma_b(X)_{|\mathscr{F}_1 \cup \mathscr{F}_2}) \longrightarrow C^*(\Gamma_b(X)_{|\mathscr{F}_2}) \longrightarrow 0 \ .$$

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We produce the 6-term exact sequence :

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The $\delta_i : K_i(C_0(B))^{\#\mathscr{F}_2} \to K_i(C_0(B))^{\#\mathscr{F}_1}$ are coordinatewise the index maps of the sequence :

$$0 \longrightarrow C^*(\Gamma_b(X)_{|g}) \longrightarrow C^*(\Gamma_b(X)_{|g\cup f}) \longrightarrow C^*(\Gamma_b(X)_{|f}) \longrightarrow 0 \;.$$

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If $\underline{f \not\subset \partial\overline{g}}$: $g \underset{B}{\times} g \times \mathbb{R}^{+*} \underset{\sim}{\sqcup} f \underset{B}{\times} f \times \mathbb{R}^{+*2} = g \underset{B}{\times} g \times \mathbb{R}^{+*} \sqcup f \underset{B}{\times} f \times \mathbb{R}^{+*2}$. Spliting then index vanishes.

Computation of $K_*(C^*(\Gamma_b(X)_{|\mathscr{F}_1 \cup \mathscr{F}_2}))$

If $\underline{f \subset \overline{g} \cap \overline{g'}}$: We have $g \subset H_i$, $g' \subset H_j$.

Let $U \subset \widetilde{X}$ a tubular neighborhood of f : $(U \cap H_k \neq \emptyset \Rightarrow k \in \{i, j\})$, $U \cap H_i \cap H_j = f$:
$U = f \times \mathbb{R}^2 \quad U_g = f \times (\mathbb{R}^{+*} \times \{0\}) \quad U_{g'} = f \times (\{0\} \times \mathbb{R}^{+*}) \quad U_f = f \times (\{0\} \times \{0\}) = f.$

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$$\begin{aligned}
\Gamma_b(X)_{|U} &= Mt_N(X)^c_{|U} = Bst^c\left(Bst^c(U \underset{B}{\times} U, U_g \underset{B}{\times} U_g), Bst^c(U_{g'} \underset{B}{\times} U_{g'}, U_f \underset{B}{\times} U_f)\right) \\
&= f \underset{B}{\times} f \times \left((\mathbb{R}^{+*2})^2 \underset{\sim}{\bigsqcup} (\mathbb{R}^{+*} \times \{0\})^2 \times \mathbb{R}^{+*} \underset{\sim}{\bigsqcup} (\{0\} \times \mathbb{R}^{+*})^2 \times \mathbb{R}^{+*} \underset{\sim}{\bigsqcup} (\{0\}^2)^2 \times \mathbb{R}^{+*2}\right) \\
&= (f \underset{B}{\times} f \times (\mathbb{R}^{+*2})^2) \underset{\sim}{\bigsqcup} \Gamma_b(X)_{|U_g} \underset{\sim}{\bigsqcup} \Gamma_b(X)_{|U_{g'}} \underset{\sim}{\bigsqcup} \Gamma_b(X)_{|f}.
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&= f \underset{B}{\times} f \times \left((\mathbb{R}^{+*2})^2 \underset{\sim}{\bigsqcup} (\mathbb{R}^{+*} \times \{0\})^2 \times \mathbb{R}^{+*} \underset{\sim}{\bigsqcup} (\{0\} \times \mathbb{R}^{+*})^2 \times \mathbb{R}^{+*} \underset{\sim}{\bigsqcup} (\{0\}^2)^2 \times \mathbb{R}^{+*2}\right)\\
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\end{aligned}$$

Which has the chart :
$$\begin{array}{lcl}
(x_1, x_2), (y_1, y_2) & \mapsto & (\frac{y_1}{x_1}, x_1, \frac{y_2}{x_2}, x_2)\\
(x_1, 0), (y_1, 0), \mu & \mapsto & (\frac{y_1}{x_1}, x_1, \mu, 0)\\
(0, x_2), (0, y_2), \lambda & \mapsto & (\lambda, 0, \frac{y_2}{x_2}, x_2)\\
(0, 0), (0, 0), \lambda, \mu & \mapsto & (\lambda, 0, \mu, 0)
\end{array}.$$

Computation of $K_*(C^*(\Gamma_b(X)_{|\mathscr{F}_1 \cup \mathscr{F}_2}))$

Setting the groupoid $\mathbb{R}^+ \rtimes_j \mathbb{R}^2$ associated to the action $\mathbb{R}^+ \curvearrowleft \mathbb{R}^2$ defined by $t * (\lambda_1, \lambda_2) := te^{\lambda_j}$, we have :
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$\Gamma_b(X)_{|f\cup U_g} \cong f \underset{B}{\times} f \times (\mathbb{R}^+ \rtimes_1 \mathbb{R}^2)$ and $\Gamma_b(X)_{|f\cup U_{g'}} \cong f \underset{B}{\times} f \times (\mathbb{R}^+ \rtimes_2 \mathbb{R}^2)$.

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$$\begin{tikzcd} 0 \arrow[r] & C^*(\Gamma_b(X)_{|U_g}) \arrow[r] & \overbrace{C^*(\Gamma_b(X)_{|U_g\cup f})}^{\cong C^*(f \underset{B}{\times} f \times (\mathbb{R}^+ \rtimes_1 \mathbb{R}^2))} \arrow[r] & C^*(\Gamma_b(X)_{|f}) \arrow[r] \arrow[ll, bend left, squiggly, "\alpha"] & 0 \end{tikzcd}$$

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$$\begin{tikzcd} 0 \arrow[r] & C^*(\Gamma_b(X)_{|U_{g'}}) \arrow[r] & \overbrace{C^*(\Gamma_b(X)_{|U_{g'}\cup f})}^{\cong C^*(f \underset{B}{\times} f \times (\mathbb{R}^+ \rtimes_2 \mathbb{R}^2))} \arrow[r] & C^*(\Gamma_b(X)_{|f}) \arrow[r] \arrow[ll, bend left, squiggly, "-\alpha"] & 0 \end{tikzcd}$$

Lemma : (proof in progress)

The maps $K_*(i) : K_*(C^*(\Gamma_b(X)_{|U_g})) \longrightarrow K_*(C^*(\Gamma_b(X)_{|g}))$ are isomorphisms.

$$\begin{tikzcd}
0 \arrow[r] & C^*(\Gamma_b(X)_{|g}) \arrow[r] & C^*(\Gamma_b(X)_{|f\cup g}) \arrow[r] & C^*(\Gamma_b(X)_{|f}) \arrow[r] & 0 \\
0 \arrow[r] & C^*(\Gamma_b(X)_{|U_g}) \arrow[r] \arrow[u, hook, "i"] & C^*(\Gamma_b(X)_{|U_g\cup f}) \arrow[r] \arrow[u, hook] & C^*(\Gamma_b(X)_{|f}) \arrow[r] \arrow[u, equal] & 0
\end{tikzcd}$$

Computation of $K_*(C^*(\Gamma_b(X)_{|\mathscr{F}_1 \cup \mathscr{F}_2}))$

Lemma : (proof in progress)

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0 \arrow[r] & C^*(\Gamma_b(X)_{|U_g}) \arrow[r] \arrow[u, hook, "i"] \arrow[d, equal] & C^*(\Gamma_b(X)_{|U_g \cup f}) \arrow[r] \arrow[u, hook] \arrow[d, equal] & C^*(\Gamma_b(X)_{|f}) \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & C^*(f \underset{B}{\times} f \times \mathbb{R}^{+*2} \times \mathbb{R}^{+*}) \arrow[r] & C^*(f \underset{B}{\times} f \times (\mathbb{R}^+ \rtimes_1 \mathbb{R}^2)) \arrow[r] & C^*(f \underset{B}{\times} f \times \mathbb{R}^{+*2}) \arrow[r] \arrow[ll, squiggly, bend left, "\alpha"] & 0.
\end{tikzcd}$$

Computation of $K_*(C^*(\Gamma_b(X)_{|\mathscr{F}_1 \cup \mathscr{F}_2}))$

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0 \arrow[r] & C^*(\Gamma_b(X)_{|U_g}) \arrow[r] \arrow[u, hook, "i"] \arrow[d, equal] & C^*(\Gamma_b(X)_{|U_g \cup f}) \arrow[r] \arrow[u, hook] \arrow[d, equal] & C^*(\Gamma_b(X)_{|f}) \arrow[r] \arrow[u, equal] \arrow[d, equal] & 0 \\
0 \arrow[r] & C^*(f \underset{B}{\times} f \times \mathbb{R}^{+*2} \times \mathbb{R}^{+*}) \arrow[r] & C^*(f \underset{B}{\times} f \times (\mathbb{R}^{+} \rtimes_1 \mathbb{R}^2)) \arrow[r] & C^*(f \underset{B}{\times} f \times \mathbb{R}^{+*2}) \arrow[r] \arrow[ll, squiggly, bend left, "\alpha"] & 0.
\end{tikzcd}
$$

Remark :

Because of $C^*(f \underset{B}{\times} f)$ nuclearity, it is enough to study the connection maps associated to

$$
0 \longrightarrow C^*(\mathbb{R}^{+*2} \times \mathbb{R}^{+*}) \longrightarrow C^*(\mathbb{R}^{+} \rtimes_1 \mathbb{R}^2) \longrightarrow C^*(\mathbb{R}^{+*2}) \longrightarrow 0 \ .
$$

Computation of $K_*(C^*(\Gamma_b(X)_{|\mathscr{F}_1 \cup \mathscr{F}_2}))$

To study K-theory of $C^*(\mathbb{R}^+ \rtimes_1 \mathbb{R}^2)$ we define $(\mathbb{R}^+ \times [0,1]) \rtimes \mathbb{R}^2$ the groupoid associated to the action $(\mathbb{R}^+ \times [0,1]) \curvearrowleft \mathbb{R}^2$ defined by $(t, \varepsilon) * (\lambda_1, \lambda_2) := (te^{\varepsilon \lambda_1}, \varepsilon)$.

Computation of $K_*(C^*(\Gamma_b(X)_{|\mathscr{F}_1\cup\mathscr{F}_2}))$

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Using 6-terms exact sequence asociated to

$$0 \longrightarrow C^*((\mathbb{R}^+ \times]0,1]) \rtimes \mathbb{R}^2) \longrightarrow C^*((\mathbb{R}^+ \times [0,1]) \rtimes \mathbb{R}^2) \xrightarrow{ev_0} C^*(\mathbb{R}^+ \times \mathbb{R}^2) \longrightarrow 0$$

$$0 \longrightarrow C^*((\mathbb{R}^+ \times [0,1[) \rtimes \mathbb{R}^2) \longrightarrow C^*((\mathbb{R}^+ \times [0,1]) \rtimes \mathbb{R}^2) \xrightarrow{ev_1} C^*(\mathbb{R}^+ \rtimes_1 \mathbb{R}^2) \longrightarrow 0.$$

Computation of $K_*(C^*(\Gamma_b(X)_{|\mathscr{F}_1 \cup \mathscr{F}_2}))$

To study K-theory of $C^*(\mathbb{R}^+ \rtimes_1 \mathbb{R}^2)$ we define $(\mathbb{R}^+ \times [0,1]) \rtimes \mathbb{R}^2$ the groupoid associated to the action $(\mathbb{R}^+ \times [0,1]) \curvearrowleft \mathbb{R}^2$ defined by $(t,\varepsilon) * (\lambda_1, \lambda_2) := (te^{\varepsilon \lambda_1}, \varepsilon)$.

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we get :

$$\begin{tikzcd}
K_*(C^*(\mathbb{R}^+ \times \mathbb{R}^2)) \arrow[d, equal] & K_*(C^*((\mathbb{R}^+ \times [0,1]) \rtimes \mathbb{R}^2)) \arrow[l, "\cong"', "ev_0"] \arrow[r, "\cong", "ev_1"'] & K_*(C^*(\mathbb{R}^+ \rtimes_1 \mathbb{R}^2)) \\
0. & &
\end{tikzcd}$$

To study K-theory of $C^*(\mathbb{R}^+ \rtimes_1 \mathbb{R}^2)$ we define $(\mathbb{R}^+ \times [0,1]) \rtimes \mathbb{R}^2$ the groupoid associated to the action $(\mathbb{R}^+ \times [0,1]) \curvearrowleft \mathbb{R}^2$ defined by $(t,\varepsilon) * (\lambda_1, \lambda_2) := (te^{\varepsilon\lambda_1}, \varepsilon)$.

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$$0 \longrightarrow C^*((\mathbb{R}^+ \times [0,1[) \rtimes \mathbb{R}^2) \longrightarrow C^*((\mathbb{R}^+ \times [0,1]) \rtimes \mathbb{R}^2) \xrightarrow{ev_1} C^*(\mathbb{R}^+ \rtimes_1 \mathbb{R}^2) \longrightarrow 0.$$

we get :

$$\begin{tikzcd}
K_*(C^*(\mathbb{R}^+ \times \mathbb{R}^2)) \arrow[d, equal] & K_*(C^*((\mathbb{R}^+ \times [0,1]) \rtimes \mathbb{R}^2)) \arrow[l, "\cong"', "ev_0"] \arrow[r, "\cong", "ev_1"'] & K_*(C^*(\mathbb{R}^+ \rtimes_1 \mathbb{R}^2)) \\
0. & &
\end{tikzcd}$$

Then

$$\begin{tikzcd}
0 \arrow[r] & C^*(\mathbb{R}^{+*2} \times \mathbb{R}^{+*}) \arrow[r] & C^*(\mathbb{R}^+ \rtimes_1 \mathbb{R}^2) \arrow[r] & C^*(\mathbb{R}^{+*2}) \arrow[r] \arrow[ll, squiggly, bend left, "\cong"] & 0 .
\end{tikzcd}$$

Eventually, α is an isomorphism.

Computation of $K_*(C^*(\Gamma_b(X)_{|\mathscr{F}_1\cup\mathscr{F}_2}))$

Index maps

$$\begin{tikzcd}
K_0(C_0(B))^{\#\mathscr{F}_1} \arrow[r] & K_1(C^*(\Gamma_b(X)_{|\mathscr{F}_1\cup\mathscr{F}_2})) \arrow[r] & K_1(C_0(B))^{\#\mathscr{F}_2} \arrow[d, "\delta_1"] \\
K_0(C_0(B))^{\#\mathscr{F}_2} \arrow[u, "\delta_0"] & K_0(C^*(\Gamma_b(X)_{|\mathscr{F}_1\cup\mathscr{F}_2})) \arrow[l] & K_1(C_0(B))^{\#\mathscr{F}_1} \arrow[l]
\end{tikzcd}$$

we have :

$$\begin{array}{cccc}
\delta_i : & K_i(C_0(B))^{\#\mathscr{F}_2} & \longrightarrow & K_i(C_0(B))^{\#\mathscr{F}_1} \\
& (0,\dots,0,\underbrace{b,}_{f \text{ coordinate}} 0,\dots,0) & \mapsto & (0,\dots,0,\underbrace{\alpha(b)}_{g \text{ coordinate}},\dots,\underbrace{-\alpha(b),}_{g' \text{ coordinate}} 0,\dots,0).
\end{array}$$

$\rightarrow$ Coherent choice of signs ? Conormal homology.

Computation of $K_*(C^*(\Gamma_b(X)_{|\mathscr{F}_1\cup\mathscr{F}_2}))$

Index maps

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K_0(C_0(B))^{\#\mathscr{F}_1} \arrow[r] & K_1(C^*(\Gamma_b(X)_{|\mathscr{F}_1\cup\mathscr{F}_2})) \arrow[r] & K_1(C_0(B))^{\#\mathscr{F}_2} \arrow[d, "\delta_1"] \\
K_0(C_0(B))^{\#\mathscr{F}_2} \arrow[u, "\delta_0"] & K_0(C^*(\Gamma_b(X)_{|\mathscr{F}_1\cup\mathscr{F}_2})) \arrow[l] & K_1(C_0(B))^{\#\mathscr{F}_1} \arrow[l]
\end{tikzcd}$$

we have :

$$\delta_i : \begin{array}{ccc} K_i(C_0(B))^{\#\mathscr{F}_2} & \longrightarrow & K_i(C_0(B))^{\#\mathscr{F}_1} \\ (0,\dots,0,\underbrace{b,}_{f\text{ coordinate}}0,\dots,0) & \mapsto & (0,\dots,0,\underbrace{\alpha(b)}_{g\text{ coordinate}},\dots,\underbrace{-\alpha(b),}_{g'\text{ coordinate}}0,\dots,0). \end{array}$$

$\rightarrow$ Coherent choice of signs ? Conormal homology.

Theorem

The obstruction $K_0(C^*(G_F)) = K_0(C^*(\Gamma_b(X)_{|\mathscr{F}_1\cup\mathscr{F}_2}))$ fits in the short exact sequence :

$$0 \longrightarrow K_1(C_0(B))^{\#\mathscr{F}_1}/Im\,\delta_1 \longrightarrow K_0(C^*(G_F)) \longrightarrow Ker\delta_0 \longrightarrow 0\ .$$

Thank you for your attention !