From geometry to K-theory through deformation groupoïds

Towards the Atiyah-Singer theorem

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Introduction

This project is situated within the framework of index theory and its developments since the 1960s. The cornerstone of this theory is the famous Atiyah-Singer theorem, which we aim to partially re-prove here using some modern tools developed since the 1980s.

Theorem (Atiyah-Singer 60's)

Let M be a compact manifold without boundary, D an elliptic differential operator. Then D is a Fredholm operator, and its index satisfies :
$Ind(D) = \langle [T^*M], ch([\sigma_D]) \wedge Td_M \rangle$.

The Atiyah-Singer theorem aims to establish a connection between algebraic properties of a differential operator, which are properties related to the solutions of a partial differential equation, and geometric properties of the considered space. The algebraic translation of different invariants led to the birth of K-theory. This is an important field of study today. It provides a new way to analyze partial differential equations. The new tools developed, such as Connes' tangent groupoid and the deformation into a normal cone, offer a new approach that allows for visualization of phenomena and facilitates the exploration of generalizations of the theorem. These generalizations were more challenging to envision using the initial, more "technical" approaches.

The global strategy of the Connes's proof by deformation groupoïds is the following one:

First:
We use groupoïds of deformation and K-theory to construct the map:

$$\begin{array}{ccccccc} K^0_{top}(T^*M) & = & K_0(C_0(T^*M)) & \xrightarrow{[K_0(\widetilde{\mathcal{F}})\circ K_0(ev_0)]^{-1}} & K_0(C^*(G_M^{tan})) & \xrightarrow{K_0(e_1)} & K_0(C^*(M \times M)) \\ & & & & & & \downarrow {\scriptstyle K_0(\widetilde{\pi_{x_0}})\ \cong} \\ & & & & & & \mathbb{Z} \end{array}$$

Second:
All the elliptic operators are Fredholm in this context.
Then we show that the index map $Ind : Ell(M) \longrightarrow \mathbb{Z}$ could be factorised through $K^0_{top}(T^*M)$ using the symbol map. Then appear a new map Ind_a^M which make this triangle commutative.

$$\begin{array}{ccc} Ell(M) & \xrightarrow{Ind} & \mathbb{Z} \\ \downarrow\!\!\downarrow & \nearrow_{Ind_a^M} & \\ K^0_{top}(T^*M) & & \end{array}$$

This map is called the analytical index.

Third:
We show that the map we construced in the first step is equal to the analytical index map. Namely $Ind_a^M = K_0(\widetilde{\pi_{x_0}}) \circ K_0(ev_1) \circ [K_0(\widetilde{\mathcal{F}}) \circ K_0(ev_0)]^{-1}$.

Fourth:
We introduce a last deformation from the analytical index map Ind_a^M, producing new K-theoretical links which will provide the right side of the desired equality.

In this document we will be focused on the first step. We will develop the tools we need to construct the map from $K^0_{top}(T^*M)$ to $\mathbb{Z}$. We will also begin to explore the concept of Morita equivalence which is a strong notion in K-theory. Even if it will not help directly to construct the map we want.

The chapter 1 will be devoted to the notion of groupoïds: In the section 1.1 we define different notions of groupoïds, we state some of their first properties and we study few examples. With section 1.2 we look at the numerical functions defined on a groupoïd to construct C^*-algebras over it and we study two examples which will be usefull at the end of this document. The section 1.3 of this chapter could be avoided for a first reading. We will not use its notions elsewhere in the document. In this section we are attached to define a new notion of equivalence between groupoïds which is more adapted to K-theoretical studies.

The chapter 2 goal is to construct a geometric deformation in terms of manifold, namely the deformation to the normal cone: We begin in section 2.1 by defining the normal bundle of a pair of manifold, and we establish some of its properties. Then in section 2.2 we use this bundle to construct the deformation to the normal cone of a pair of manifold. We show some of its properties, especially functoriality. We finish this chapter with section 2.3 where we compute explicitly the charts of a deformation to the normal cone on an easy example.

In the chapter 3 we encode the deformation to the normal code using groupoïds, and we use K-theoretical properties to finally construct the map $K_0(\widetilde{\pi_{x_0}}) \circ K_0(ev_1) \circ [K_0(\widetilde{\mathcal{F}}) \circ K_0(ev_0)]^{-1}$: In section 3.1 we translate the deformation between manifolds in a deformation between groupoïds by defining the tangent groupoïd of a Lie groupoïd. Then we establish a link between the involved C^*-algebras. And we exploit these links using K-theory. Finally in section 3.2 we use these links on a specific example which involve the *Connes's tangent groupoïd*. Then we use more K-theory to get the map we want to accomplish the first step of the Atiyah-Singer theorem proof.

Chapter 1

Groupoïds

1.1 Definition and example

Definition 1.1.1 (Groupoïds)

A groupoïd denoted $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$, or only $\mathcal{G}$, is a small category in which all morphisms are invertible. Formally $\mathcal{G}$ (resp. $\mathcal{G}^{(0)}$) denote the set of morphisms (resp. the set of objects) of the category. This category is endowed with maps coming from the categorical structure:

- Two maps $s, t : \mathcal{G} \to \mathcal{G}^{(0)}$ called respectively source and target which maps a morphism to its source (resp. target) object.
- A product map $m : \mathcal{G}_2 \to \mathcal{G}$ called *composition*, where $\mathcal{G}_2 = \{(g,h) \in \mathcal{G} \times \mathcal{G} : t(h) = s(g)\}$. Elements of such a couple are called to be *composable* and their product will be denoted gh.
- An application $u : \mathcal{G}^{(0)} \to \mathcal{G}$ which maps an object x to its identity map $\mathbb{1}_x$.
- An application $i : \mathcal{G} \to \mathcal{G}$ which maps an arrow to its inverse.

Notation : For $x \in \mathcal{G}^{(0)}$ we denote by $\mathcal{G}^x$ (resp. $\mathcal{G}_x$) the $t - fiber$ (resp. $s - fiber$) associated to x. We also call *orbit of* x the subset $t(\mathcal{G}_x)$ and we denote it $\mathcal{O}_x$.

Remark 1.1.2

The following facts are checked:

- $s(gh) = s(h)$, $t(gh) = t(g) \quad \forall (g,h) \in \mathcal{G}_2$
- $s(g) = t(g^{-1}) \quad \forall g \in \mathcal{G}$
- $(gh)k = g(hk) \quad \forall (g,h),(h,k) \in \mathcal{G}_2$
- $\mathbb{1}_{t(g)} g = g = g \mathbb{1}_{s(g)} \quad \forall g \in \mathcal{G}$

Remark 1.1.3

Conversly, finding sets $\mathcal{G}$ and $\mathcal{G}^{(0)}$ and maps s, t, m, u, i satisfying these relations is enough to define a groupoïd.

We will define our groupoïds using arbitrarily the definition or the last remark.

This purely algebraic structure is a generalization of multiples common ones (sets, groups, ...).

Example 1.1.4

To a set X we can associate the trivial groupoïd $\{\mathbb{1}_x : x \in X\} \rightrightarrows X$ defined by the trivial category associated to the set X.

Example 1.1.5

Every group G define a groupoïd with one object $*$ by fixing the category $\mathcal{G}$:

$$\begin{cases} Obj = \{*\} \\ Mor_{\mathcal{G}}(*, *) = G \end{cases}$$

In these two cases, maps are easily defined. Actually these groupoïds are special cases of *action groupoïds*:

Example 1.1.6 (action groupoïd)

From a group G acting on a set X, $\begin{matrix} G \times X & \to & X \\ (g, x) & \mapsto & g.x \end{matrix}$. We can define the action groupoïd $X \rtimes G \rightrightarrows X$ by setting the category

$$\begin{cases} Obj = X \\ Mor(x, y) = \{g \in G : g.x = y\} \end{cases}$$

We can also construct an action groupoïd by defining the maps s, t, m, u, i.

Example 1.1.7 (action groupoïd - 2^{nd} construction)

As previously we take an action $\begin{matrix} G \times X & \to & X \\ (g, x) & \mapsto & g.x \end{matrix}$, and we define the groupoïd $X \times G \rightrightarrows X$ by setting:

- $s(x, g) = x$
- $t(x, g) = g.x$
- $m((hy, g), (y, h)) = (y, gh)$
- $u(x) = (x, e_G)$
- $i(x, g) = (gx, g^{-1})$

More generally we can construct a groupoïd associated to an equivalence relation.

Example 1.1.8 (Equivalence relation groupoïd)

If a set X is endowed with an equivalence relation $\sim$, we recall that its graph is defined by $Graph(\sim) = \{(x, y) \in X \times X : x \sim y\} \subset X \times X$.
Then we define the associated groupoïd by the category:

$$\begin{cases} Obj = X \\ Mor = Graph(\sim) \end{cases}$$

denoted $Graph(\sim) \rightrightarrows X$.
We should mention the two extremal cases:

- If $\sim$ is defined by $x \sim y$ if and only if $x = y$, then we get the diagonal groupoïd $\Delta_X \rightrightarrows X$
- If $x \sim y$ for all x, y in X, then we get the pair groupoïd $X \times X \rightrightarrows X$.

Even if this groupoïd look really simple, it will play a great role in the next pages.

Definition 1.1.9 ((*strict*) Groupoïd morphism)

A morphism F from a groupoïd $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ to another $\mathcal{H} \rightrightarrows \mathcal{H}^{(0)}$ is a functor between underlying categories.
More explicitely, it is a pair of maps $(F : \mathcal{G} \longrightarrow \mathcal{H},\ f : \mathcal{G}^{(0)} \longrightarrow \mathcal{H}^{(0)})$ such that:

- $\forall (g : x \to y) \in \mathcal{G},\ (F(g) : f(x) \to f(y)) \in \mathcal{H}$
- $\forall (g, h) \in \mathcal{G}_2,\ F(gh) = F(g)F(h)$
- $\forall x \in \mathcal{G}^{(0)},\quad F(\mathbb{1}_x) = \mathbb{1}_{f(x)}$
- $\forall (g : x \to y) \in \mathcal{G},\quad F(g^{-1}) = F(g)^{-1}$ (here it is a consequence of the two previous statements)

These four conditions are equivalent to the commutation of five diagrams (one for each structural map (s, t, m, u, i). Usually to denote it we only write:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{F} & \mathcal{H} \\ \downdownarrows & & \downdownarrows \\ \mathcal{G}^{(0)} & \xrightarrow{f} & \mathcal{H}^{(0)} \end{array} \quad \text{or just } \mathcal{G} \xrightarrow{F} \mathcal{H} .$$

Definition 1.1.10 (Sub groupoïd)

In the previous definition, if F and f are injective, then $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ is said to be a sub groupoïd of $\mathcal{H} \rightrightarrows \mathcal{H}^{(0)}$.

It is tempting to define a notion of isomorphism from this definition. But we shall see that this definition is too restrictive for our use of K-theory.

We will speak of strict morphisms to refer to the notion defined above, and not to confuse it with the notion of Hilsum-Skandalis morphisms which we will define later.

Definition 1.1.11 (topological groupoïd)

A groupoïd $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ is said to be topological when $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are topological spaces, $\mathcal{G}^{(0)}$ is Hausdorff, s, t, m, u are continuous with these topologies and i is an homeomorphism.

In many natural examples, $\mathcal{G}$ is not Hausdorff without causing any problem, that is why we only require $\mathcal{G}^{(0)}$ to be Hausdorff.

When we will have to integrate functions on our spaces, we will usually need local compacity properties. In these cases we will using the following kind of groupoïd:

Definition 1.1.12 (locally compact groupoïd)

A groupoïd $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ is said to be locally compact when it is a topological groupoïd, each element of $\mathcal{G}$ has a compact neighborhood (Hausdorff), and s is open.

Property 1.1.13

For each locally compact groupoïd, the target map t is also open.

Proof

$t = s \circ i$, s is open, i is an homeomorphism, then t is open.

Definition 1.1.14 (Lie Groupoïds)

A groupoïd $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ is said to be a Lie groupoïd if $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are smooth manifolds, $\mathcal{G}^{(0)}$ is Hausdorff, s, t, m, u are smooth, i is a diffeomorphism, s a submersion.

Remark 1.1.15

A Lie groupoïd is a locally compact groupoïd because $\mathbb{R}^n$ is locally compact, $t = s \circ i$ then t is a submsersion too, and a submersion is open.

These hypothesis gives us some structural properties.

Property 1.1.16

Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ a Lie groupoïd. For $x \in \mathcal{G}^{(0)}$ we have:

- $\mathcal{G}_x$ and $\mathcal{G}^x$ are submanifolds of $\mathcal{G}$
- $\mathcal{G}_x^x = \mathcal{G}_x \cap \mathcal{G}^x$ is a Lie group
- The orbit $\mathcal{O}_x$ is a submanifold of $\mathcal{G}^{(0)}$
- $u : \mathcal{G}^{(0)} \longrightarrow \mathcal{G}$ is an embedding of $\mathcal{G}^{(0)}$ in $\mathcal{G}$.

Proof

- $\mathcal{G}_x$ and $\mathcal{G}^x$ are fibers of submersions, then they are submanifolds of $\mathcal{G}$.
- $\mathcal{G}_x^x$ is a submanifold beacuse it is the intersection of two submanifolds of $\mathcal{G}$. We endow it with the restriction of m and i to $\mathcal{G}_x^x$.
- $\mathcal{O}_x$ is the image of $\mathcal{G}_x$ by the target map, then we can get the factorization
$$\begin{array}{ccc} \mathcal{G}_x & \xrightarrow{t|_{\mathcal{G}^x}} & \mathcal{O}_x \\ \downarrow & \nearrow \cong & \\ \mathcal{G}_x/\mathcal{G}_x^x & & \end{array}$$
. The action of $\mathcal{G}_x^x$ on $\mathcal{G}_x$ is free and proper (translation action). Then the quotient is a manifold, and then so is $\mathcal{O}_x$.
- It is enough to differentiate the identity $s \circ u = Id_{\mathcal{G}^{(0)}}$ to check that the

differential map of u is injective. Moreover it is an homeomorphism on its image, then u is an embedding.

Example 1.1.17 (Tangent bundle groupoïd)

If M is a manifold, then we define $TM \rightrightarrows M$, which will be a Lie Groupoïd, by setting for $x \in M$, $v, w \in T_x M$:
$s(x,v) = t(x,v) = pr_1(x,v) = x$; $(x,v)(x,w) = (x, v+w)$; $i(x,v) = (x,-v)$; $u(x) = (x, 0_{T_x M})$.
In this case we have $\mathcal{G}_x = \mathcal{G}^x = \mathcal{G}_x^x = T_x M$ (Lie group which is a submanifold of TM) ; $\mathcal{O}_x = \{x\}$ is a submanifold of M. And $\begin{array}{ccc} M & \hookrightarrow & TM \\ x & \mapsto & (x, 0_{T_x M}) \end{array}$ is an embedding of M in TM.

Example 1.1.18 (Action groupoïd: rotation on the circle)

We set for all this example $\theta \in \mathbb{R} \setminus \pi\mathbb{Q}$. Our concern is the action of $\mathbb{Z}$ on $\mathbb{S}^1$ by rotations of angle θ. $\begin{array}{ccc} \mathbb{Z} \times \mathbb{S}^1 & \longrightarrow & \mathbb{S}^1 \\ (n,z) & \mapsto & ze^{i\theta n} \end{array}$
This example illustrate how orbits of a simple action could arise pathological spaces, which is one justification for the use of action groupoïds.

Indeed, let's see the quotient of $\mathbb{S}^1$ by the orbits of this action : $\mathbb{S}^1/\mathcal{O}$ endowned with quotient topology.
This space is a non Hausdorff one, actually its topology is the trivial one:
Let V a non empty open subset of $\mathbb{S}^1/\mathcal{O}$, then if π is the canonical projection on the quotient, $\pi^{-1}(V)$ is open in $\mathbb{S}^1$. But because θ is not a ratio of π, then the orbits $\mathcal{O}_z$ are dense in the circle. Then for each $z \in \mathbb{S}^1$, $\mathcal{O}_z$ intersects $\pi^{-1}(V)$ and then we could find $x \in \mathbb{S}^1$ such that $\pi(z) = \pi(x)$ and $\pi(x) \in V$. In conclusion $\mathbb{S}^1/\mathcal{O} = \pi(\mathbb{S}^1) \subset V$ and then $\mathbb{S}^1/\mathcal{O} = V$.

The other approach is to avoid the quotient and to study the groupoïd associated to the action.
$\mathbb{S}^1 \times \mathbb{Z} \rightrightarrows \mathbb{S}^1$ is defined for $z \in \mathbb{S}^1$, $n, m \in \mathbb{Z}$ by:
$s(z,n) = z$; $t(z,n) = e^{i\theta n}$; $(e^{i\theta n}z, m)(z,m) = (z, m+n)$; $i(z,n) = (e^{i\theta n}z, -n)$; $u(z) = (z,0)$.
In this case we have $\mathcal{G}_z = \{(z,n) : n \in \mathbb{Z}\} \cong \mathbb{Z}$; $\mathcal{G}^z = \{(e^{-i\theta n}z, n) : n \in \mathbb{Z}\} \cong \mathbb{Z}$; $\mathcal{G}_z^z = \{(z,s)\}$ because θ is not a π ratio ; $\mathcal{O}_z = \{ze^{i\theta n} : n \in \mathbb{Z}\}$ countable and dense.

Even if our point is the study of Lie groupoïds, a lot of our results will remain true on topological and locally compact groupoïds. That is why we will state them on these more generals kind of groupoïds when it is possible to do so.

1.2 C^*- completion of a groupoïd

For a topological groupoïd $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$, we denote by $\mathcal{C}_c(\mathcal{G})$ the vector space whose elements are continuous functions from $\mathcal{G}$ to $\mathbb{C}$ compactly supported.

Definition 1.2.1 (Haar system)

A Haar system over a locally compact groupoïd $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ is a family of measures $(\nu^x)_{x\in\mathcal{G}^{(0)}}$ on the fibers $\mathcal{G}^x$, finite over compact sets such that the following two axioms holds:

- Left invariance: $\forall \gamma \in \mathcal{G}$, $\forall A \subset \mathcal{G}^{s(\gamma)}$, $\nu^{s(\gamma)}(A) = \nu^{t(\gamma)}(L_\gamma(A))$ with

$$L_\gamma : \begin{array}{ccc} \mathcal{G}^{s(\gamma)} & \longrightarrow & \mathcal{G}^{t(\gamma)} \\ \alpha & \mapsto & \gamma\alpha \end{array}, \text{ the left translation map.}$$

- Continuity: $\forall f \in \mathcal{C}_c(\mathcal{G})$, $\begin{array}{ccc} \mathcal{G}^{(0)} & \longrightarrow & \mathbb{C} \\ x & \mapsto & \int_{\mathcal{G}^x} f d\nu^x \end{array}$ is continuous.

Remark 1.2.2

By the left invariance of the measure, we can verify that we have :

$$\forall\, \gamma \in \mathcal{G},\ \forall\, f \text{ simple function, } \int_{\mathcal{G}^{s(\gamma)}} f \ \mathrm{d}\nu^{s(\gamma)} = \int_{\mathcal{G}^{t(\gamma)}} (f \circ L_{\gamma^{-1}}) \ \mathrm{d}\nu^{t(\gamma)}.$$

We can then show by density that we have this equality for continuous functions with compact support, L^2 functions, etc.

We now fix a locally compact topological groupoid $\mathcal{G}$ endowed with a Haar system.

Definition 1.2.3

Let $f, g \in \mathcal{C}_c(\mathcal{G})$. We define the convolution of f and g, by the map (when it is defined) :

$$f * g : \mathcal{G} \ni \gamma \longmapsto \int_{\mathcal{G}^{t(\gamma)}} f(\alpha)g(\alpha^{-1}\gamma) \ \mathrm{d}\nu^{t(\gamma)}(\alpha).$$

Property 1.2.4

Let $f, g \in \mathcal{C}_c(\mathcal{G})$. Then, we have :

1. $f * g \in \mathcal{C}_c(\mathcal{G})$;
2. associative property : $(f * g) * h = f * (g * h)$.

Proof

1. The continuity of $f * g$ is obtained using continuous dependency of an integral on a parameter theorem. Plus, the support of $f * g$ is also compact.
2. Via translation invariance and Fubini's theorem, we have :

$$\begin{aligned}
[(f*g)*h](\gamma) &= \int_{\mathcal{G}^{t(\gamma)}} (f*g)(\alpha)h(\alpha^{-1}\gamma)\ \mathrm{d}\nu^{t(\gamma)}(\alpha) \\
&= \int_{\mathcal{G}^{t(\gamma)}} \int_{\mathcal{G}^{t(\gamma)}} f(\tau)g(\tau^{-1}\alpha)h(\alpha^{-1}\gamma)\ \mathrm{d}\nu^{t(\gamma)}(\tau)\ \mathrm{d}\nu^{t(\gamma)}(\alpha) \\
&= \int_{\mathcal{G}^{t(\gamma)}} f(\tau) \left[\int_{\mathcal{G}^{t(\gamma)}} g(\tau^{-1}\alpha)h(\alpha^{-1}\gamma)\ \mathrm{d}\nu^{t(\gamma)}(\alpha)\right]\ \mathrm{d}\nu^{\tau(\gamma)}(\tau).
\end{aligned}$$

We set : $I(\tau) = \int_{\mathcal{G}^{t(\gamma)}} g(\tau^{-1}\alpha)h(\alpha^{-1}\gamma)\ \mathrm{d}\nu^{t(\gamma)}(\alpha)$. We also have $t(\gamma) = t(\alpha) = t(\tau)$. Then, by taking $s(\tau) = t(\tau^{-1}\gamma)$, we obtain :

$$\begin{aligned}
I(\tau) &= \int_{\mathcal{G}^{t(\gamma)}} \left[g(\cdot) \times h(\cdot^{-1}(\tau^{-1}\gamma))\right] \circ L_{\tau^{-1}}(\alpha)\ \mathrm{d}\nu^{t(\tau)}(\alpha) \\
&\underset{inv.}{=} \int_{\mathcal{G}^{s(\gamma)}} \left[g(\cdot) \times h(\cdot^{-1}(\tau^{-1}\gamma))\right](\alpha)\ \mathrm{d}\nu^{s(\tau)}(\alpha) \\
&= \int_{\mathcal{G}^{t(\tau^{-1}\gamma)}} g(\alpha)h(\alpha^{-1}\tau^{-1}\gamma)\ \mathrm{d}\nu^{t(\tau^{-1}\gamma)}(\alpha) \\
&= (g*h)(\tau^{-1}\gamma).
\end{aligned}$$

Thus, we have :

$$[(f*g)*h](\gamma) = \int_{\mathcal{G}^{t(\gamma)}} f(\tau)(g*h)(\tau^{-1}\gamma)\ \mathrm{d}\nu^{t(\gamma)}(\tau) = [f*(g*h)](\gamma).$$

Property 1.2.5

The vector space $\mathcal{C}_c(\mathcal{G})$ endowed with convolution product and the involution defined by $f \mapsto (f^* : \gamma \mapsto \overline{f(\gamma^{-1})})$ is an $*$-algebra.

From a locally compact groupoid and a choice of an Haar system, we built an algebraic object: the $*$-algebra $\mathcal{C}_c(\mathcal{G})$. Now our goal is to incorporate some analytic taste to our object by building a C^*algebra from it. Then we will be able to use K-theory on it.

We begin by setting a first norm on $\mathcal{C}_c(\mathcal{G})$:

Definition 1.2.6 (Norm $\|\cdot\|_1$)

For $f \in \mathcal{C}_c(\mathcal{G})$, we set

$$\|f\|_1 := \sup_{x\in\mathcal{G}^{(0)}} \max\left(\int_{\mathcal{G}^x} |f(\gamma)| d\nu^x(\gamma),\ \int_{\mathcal{G}^x} |f(\gamma^{-1})| d\nu^x(\gamma)\right)$$

This quantity is finite and define a norm on $\mathcal{C}_c(\mathcal{G})$.

Definition 1.2.7

We define the groupoid's full C^* algebra of $\mathcal{G}$, denoted by $C^*(\mathcal{G},\nu)$ or $C^*(\mathcal{G})$, to be the completion $\overline{\mathcal{C}_c(\mathcal{G})}^{\|\cdot\|_1}$.

In practice, this C^*-algebra can be difficult to compute, especially the norm of the different elements. For this, we will mainly focus on another completion of $\mathcal{C}_c(\mathcal{G})$, called reduced. This will be sufficient for our study ; moreover, these two C^*-algebras coincide in many cases. To do this, to each element f of $\mathcal{C}_c(\mathcal{G})$, we will associate a family of representations.

Definition 1.2.8

Let $f \in \mathcal{C}_c(\mathcal{G})$. For $x \in \mathcal{G}^{(0)}$, we define :

$$\begin{array}{rcccc} \pi_x(f) & : & \mathrm{L}^2(\mathcal{G}^x,\nu^x) & \longrightarrow & \mathrm{L}^2(\mathcal{G}^x,\nu^x) \\ & & g & \mapsto & \left[\gamma \longmapsto \int_{\mathcal{G}^{t(\gamma)}} g(\alpha) f(\alpha^{-1}\gamma)\ \mathrm{d}\nu^{t(\gamma)}(\alpha)\right] \end{array}$$

and we will denote this one by :

$$\begin{array}{rcccc} \pi_x(f) & : & \mathrm{L}^2(\mathcal{G}^x,\nu^x) & \longrightarrow & \mathrm{L}^2(\mathcal{G}^x,\nu^x) \\ & & g & \mapsto & g * f \end{array},$$

as it is an extension of the convolution operation from $\mathcal{C}_c(\mathcal{G})$ to another domain.

Property 1.2.9

In the previous definition, $\pi_x(f)$ is well defined, it is a continuous operator, and its operator norm is such that $\|\pi_x(f)\| \leqslant \|f\|_1$.

Proof

We will show all these points at the same time, by getting an inequality. Let $f \in \mathcal{C}_c(\mathcal{G})$, $x \in \mathcal{G}^{(0)}$.
For $g \in L^2(\mathcal{G}^x,\nu^x)$, for $\gamma \in \mathcal{G}^x$:

$$\begin{aligned} |g * f(\gamma)| &= \left|\int_{\mathcal{G}^x} g(\alpha) f(\alpha^{-1}\gamma) d\nu^x(\alpha)\right| \\ &\leqslant \int_{\mathcal{G}^x} |g(\alpha)||f(\alpha^{-1}\gamma)|^{1/2}|f(\alpha^{-1}\gamma)|^{1/2} d\nu^x(\alpha) \\ &\leqslant \left(\int_{\mathcal{G}^x} |g(\alpha)|^2 |f(\alpha^{-1}\gamma)| d\nu^x(\alpha)\right)^{1/2} \left(\int_{\mathcal{G}^x} |f(\alpha^{-1}\gamma)| d\nu^x(\alpha)\right)^{1/2} (C.S.I). \end{aligned}$$

But using translation we have :

$$\begin{aligned} \int_{\mathcal{G}^x} |f(\alpha^{-1}\gamma)| d\nu^x(\alpha) &= \int_{\mathcal{G}^{t(\gamma)}} |f((\gamma^{-1}\alpha)^{-1})| d\nu^{t(\gamma)}(\alpha) \\ &= \int_{\mathcal{G}^{s(\gamma)}} |f(\alpha^{-1})| d\nu^{s(\gamma)}(\alpha) \\ &= \int_{\mathcal{G}^{t(\gamma^{-1})}} |f(\alpha^{-1})| d\nu^{t(\gamma^{-1})}(\alpha) \leqslant \|f\|_1. \end{aligned}$$

Then, $\forall \gamma \in \mathcal{G}^x$,

$$|g * f(\gamma)| \leqslant \left(\int_{\mathcal{G}^x} |g(\alpha)|^2 |f(\alpha^{-1}\gamma)| d\nu^x(\alpha) \right)^{1/2} \|f\|_1^{1/2}$$

Wa are now able to check that the map $\gamma \mapsto g * f(\gamma)$ has its square modulus integrable on $\mathcal{G}^x$:

$$\begin{aligned}
\int_{\mathcal{G}^x} |g * f(\gamma)|^2 d\nu^x(\gamma) &\leqslant \|f\|_1 \int_{\mathcal{G}^x} \int_{\mathcal{G}^x} |g(\alpha)|^2 |f(\alpha^{-1}\gamma)| d\nu^x(\alpha) d\nu^x(\gamma) \\
&= \|f\|_1 \int_{\mathcal{G}^x} |g(\alpha)|^2 \underbrace{\left(\int_{\mathcal{G}^x} f(\alpha^{-1}\gamma)| d\nu^x(\gamma) \right)}_{\leqslant \|f\|_1} d\nu^x(\alpha) \ (Fubini) \\
&\leqslant \|f\|_1^2 \, \|g\|_2^2.
\end{aligned}$$

Then $g * f$ has its square modulus integrable, and then $\pi_x(f)$ is well defined. Moreover $\|\pi_x(f)(g)\|_2 \leqslant \|f\|_1 \|g\|_2$, as $\pi_x(f)$ is linear, it is a bounded operator and $\|\pi_x(f)\| \leqslant \|f\|_1$.

Then the objects $\{\pi_x\}_{x \in \mathcal{G}^{(0)}}$ are representations of $\mathcal{C}_c(\mathcal{G})$ in the Hilbert space $L^2(\mathcal{G}^x, \nu^x)$.

Definition 1.2.10 Reduced C^*-algebra and reduced norm

For $f \in \mathcal{C}_c(\mathcal{G})$, we set $\|f\|_r = \sup_{x \in \mathcal{G}^{(0)}} \|\pi_x(f)\|$. It defines a norm $\|.\|_r$ called the reduced norm.
Using this norm we define the C^*-algebra $\overline{\mathcal{C}_c(\mathcal{G})}^{\|\cdot\|_r}$ called *reduced C^*-algebra of* $\mathcal{G}$, denoted $C_r^*(\mathcal{G})$.

Now we will see some examples of explicit computations of reduced algebras.

Example 1.2.11 Pair of manifold

Let M a manifold with a measure ν. We take the pair groupoïd $M \times M \rightrightarrows M$ that we met in the previous section.
Then in this case, the translated meaures ν^x defined on $\{x\} \times M = \mathcal{G}^x$ by $\nu^x(\{x\} \times A) = \nu(A)$, for $x \in M$ and A any measurable subset of M, they define a Haar system on our groupoïd.

To deal with this example we will admit two lemmas: for $x_0 \in M$,

- $\forall f \in \mathcal{C}_c(M \times M)$, $\pi_{x_0}(f) \in \mathcal{K}(L^2(\{x_0\} \times M))$
- $\overline{\{\pi_{x_0}(f) : f \in \mathcal{C}_c(M \times M)\}}^{\|\cdot\|_{B(L^2(\{x_0\} \times M))}} = \mathcal{K}(L^2(\{x_0\} \times M))$

.
Let $x, z \in M$: Using the left invariance of the measure, we have that the left translation maps $\begin{array}{cccc} L_{(x,z)} : & L^2(\{x\} \times M) & \longrightarrow & L^2(\{z\} \times M) \\ & g & \mapsto & [(z,y) \mapsto g((x,z)(z,y)) = g(x,y)] \end{array}$ is an isometry. In particular its norm is equal to 1.
We can also use this invariance to show that $\pi_x(f) = L_{(x,z)} \circ \pi_z(f) \circ L_{(z,x)}$. Then we have $\|\pi_x(f)\| = \|\pi_z(f)\|$.

Then we have $\|f\|_r = \sup_{x\in M} \|\pi_x(f)\| = \|\pi_{x_0}(f)\|$, with $x_0 \in M$.
We fix such a $x_0 \in M$, then we set the map:

$$\begin{array}{cccc} \pi_{x_0}: & (\mathcal{C}_c(M\times M), \|.\|_r) & \longrightarrow & (\mathcal{K}(L^2(\{x_0\}\times M)), \|.\|) \\ & f & \mapsto & \pi_{x_0}(f) \end{array}$$

, which is well defined because of the first lemma.
The space $\mathcal{K}(L^2(\{x_0\}\times M))$ is closed (as the closure of finite rank operators) in $B(L^2(\{x_0\}\times M))$ which is a complete space. Then $\mathcal{K}(L^2(\{x_0\}\times M))$ is complete.
Moreover this map is continuous because it is an isometry.
Then we can use the extension theorem proved in appendix, to get a continuous linear extension:

$$\widetilde{\pi_{x_0}}: \quad C_r^*(M\times M) \longrightarrow \mathcal{K}(L^2(\{x_0\}\times M))$$

If $f \in C_r^*(M\times M)$, then we have $(f_n)_n$ a sequence of $\mathcal{C}_c(M\times M)$ such that $f_n \underset{n\to+\infty}{\longrightarrow} f$.
Then putting $\|f_n\|_r = \|\pi_{x_0}(f_n)\|$ to the limit, we have $\|f\|_r = \|\pi_{x_0}(f)\|$, then $\widetilde{\pi_{x_0}}$ is still an isometry. In particular it is an injective map.
Let $k \in \mathcal{K}(L^2(\{x_0\}\times M))$, then using the second lemma we get a sequence $(f_n)_n$ of $\mathcal{C}_c(M\times M)$ such that $\pi_{x_0}(f_n) \underset{n\to+\infty}{\longrightarrow} k$. Then $(f_n)_n$ is a Cauchy sequence, and $C_r^*(M\times M)$ is complete as a completion of $\mathcal{C}_c(M\times M)$. Then we have $f \in C_r^*(M\times M)$ such that $f_n \underset{n\to+\infty}{\longrightarrow} f$. Then following the construction of the extension $\widetilde{\pi_{x_0}}$, we have :
$\widetilde{\pi_{x_0}}(f) = \lim_{n\to+\infty} \pi_{x_0}(f_n) = k$.
Then $\widetilde{\pi_{x_0}}$ is surjective.

Then we have an isomorphism : $C_r^*(M\times M) \cong \mathcal{K}(L^2(\{x_0\}\times M)) \cong \mathcal{K}(L^2(M))$.

Example 1.2.12 Riemannian vector bundle

Let $p: E \to M$ a Riemannian vector bundle above a manifold. For $x \in M$, we denote by E_x the fiber $p^{-1}(x)$.
This bundle define a groupoïd $\mathcal{G}$ with: $s = t = p$; $(x,v).(x,w) = (x, v+w)$; $i(x,v) = (x,-v)$; $u(x) = (x, 0_{E_x})$.

Our goal is to show that $C_r^*(E) \cong C_0(E^*)$.
We define the fiberwise Fourier transform as: $\forall x \in M$

$$\mathcal{F}|_{E_x} = \mathcal{F}_x : \begin{array}{ccc} L^2(E_x) & \overset{\cong \text{ isometry}}{\longrightarrow} & L^2(E_x^*) \\ g & \mapsto & \left(\varphi \mapsto \frac{1}{(2\pi)^{n/2}} \int_{E_x} e^{-i\varphi(v)} g(v)dv\right) \end{array}$$

which is, according to Fourier theory, a bijective isometry.
Moreover if we canonically identify E_x and its bidual, then $\mathcal{F}_x \circ \mathcal{F}_x(g) = g \circ \sigma$ where $\sigma = -Id_{\mathbb{R}^n}$.
We also know that the Fourier transform convert a convolution product in a usual product : $\forall f, g \in L^2(E_x), \mathcal{F}_x(f * g) = (2\pi)^{n/2}\mathcal{F}_x(f)\mathcal{F}_x(g)$.
We will apply these properties fiberwise, then for $f \in \mathcal{C}_c(\mathcal{G})$, $g \in L^2(E_x^*)$, we get:

$$(\mathcal{F}_x \circ \pi_x(f) \circ \mathcal{F}_x^{-1})(g) = \mathcal{F}_x(\pi_x(f)(\mathcal{F}_x^{-1}(g))) = \mathcal{F}_x((\mathcal{F}_x \circ \sigma(g)) * f) = (2\pi)^{n/2} g \mathcal{F}_x(f)$$

Then the operator $\mathcal{F}_x \circ \pi_x(f) \circ \mathcal{F}_x^{-1}$ is the operator "multiplication by $(2\pi)^{n/2}\mathcal{F}_x(f)$". But the function $\mathcal{F}_x(f)$ is bounded, and we know the subordinate norm of such operators $\|\mathcal{F}_x \circ \pi_x(f) \circ \mathcal{F}_x^{-1}\| = \|(2\pi)^{n/2}\mathcal{F}_x(f)\|_{\infty, E_x^*}$.

Then for $f \in \mathcal{C}_c(\mathcal{G})$ we have :

$$\|f\|_r = \sup_{x\in M}\|\pi_x(f)\| = \sup_{x\in M}\|\mathcal{F}_x \circ \pi_x(f) \circ \mathcal{F}_x^{-1}\| = (2\pi)^{n/2}\sup_{x\in M}\|\mathcal{F}_x(f)\|_{\infty,E_x^*} = (2\pi)^{n/2}\|\mathcal{F}(f)\|_{\infty,E^*}.$$

Then we can define: $\begin{array}{cccc} \mathcal{F}: & (\mathcal{C}_c(E), \|.\|_r) & \longrightarrow & (C_0(E^*), \|.\|_{\infty,E^*}) \\ & f & \mapsto & \mathcal{F}(f) \end{array}$ which is continuous because $\|\mathcal{F}(f)\|_{\infty,E^*} = \frac{1}{(2\pi)^{n/2}}\|f\|_r$.
The normed vector space $(C_0(E^*), \|.\|_{\infty,E^*})$ is complete, then by extension theorem of the appendix, we can extend $\mathcal{F}$ as $\widetilde{\mathcal{F}} : C_r^*(E) \longrightarrow C_0(E^*)$.
With the same argument we used with $\widetilde{\pi_{x_0}}$, we have $\|\widetilde{\mathcal{F}}(f)\|_{\infty,E^*} = \frac{1}{(2\pi)^{n/2}}\|f\|_r$, then $\widetilde{\mathcal{F}}$ is injective.

Using Fourier theory on vector bundles, we can show that the image of $\widetilde{\mathcal{F}}$ contains a class of functions which is dense in $(C_0(E^*), \|.\|_\infty)$. Then using the closeness of the image, we can show that this map is sujective, then is an isomorphism.

This allows us to conclude that $C_r^*(E) \cong C_0(E)$.

1.3 Morita Equivalence

This section is isolated from the others. None of the notions, and none of the result of this section will be used in another one. Also this section is more technical and can be avoided for a first reading.

As we have seen, many mathematical objects can be written in terms of groupoids, in particular topological spaces with a topological subspace. Or manifold with a submanifold. Let's discuss this case in particular.

One way of constructing a differential manifold is to start from a topological space with an open covering. This produces a topological information. Then we define maps to form an atlas, which is part of the differential structure. We then have a natural equivalence relation through the existence of diffeomorphisms. The existence of a diffeomorphism means that two manifolds have the same differential structure and consistent topological structures with respect to the underlying topological spaces, as well as the open coverings chosen for each of them.

The development that follows is rooted in the fact that the K-theoretic properties we seek to express depend in the first place on the underlying topological space and its possible open coverings.

We shall therefore concentrate on defining a weaker notion of equivalence between groupoids which generalises the classical notion of groupoids isomorphisms in the sense of functors. This is Morita's equivalence.

Let's start by defining the terms.

Throughout this section, groupoids will be assumed to be topological unless further details are given.

Definition 1.3.1 Generalised covering

We call a generalised covering of a topological space $(X, \mathcal{O}(X))$, an application $\mathcal{U} : I \longrightarrow \mathcal{O}(X)$ from a set I to the topology of X, such that $X = \bigcup_{i \in I} \mathcal{U}(i)$.

In practice, the image of $i \in I$ by $\mathcal{U}$ will be denoted $\mathcal{U}_i$ instead of $\mathcal{U}(i)$, in the same way as the generalised covering will sometimes be denoted $\mathcal{U}$ or $\{\mathcal{U}_i\}_{i \in I}$, or even$\{\mathcal{U}_i\}_i$ if there is no risk of confusion about the set I.

Intuitively, a generalised covering is a covering in which the same set can be counted several times.

Definition 1.3.2 Disjoint union of generalised coverings

The disjoint union (or sum) of generalised coverings is defined as follows: if $(X, \mathcal{O}(X))$ is a topological space, $\mathcal{U} : I \longrightarrow \mathcal{O}(X)$ and $\mathcal{V} : J \longrightarrow \mathcal{O}(X)$ are two generalised coverings, then the application :

$$\begin{array}{ccc} I \sqcup J & \longrightarrow & \mathcal{O}(X) \\ k & \longmapsto & \begin{cases} \mathcal{U}(k) & \text{if } k \in I \\ \mathcal{V}(k) & \text{if } k \in J \end{cases} \end{array}.$$

is still a generalised covering, which we denote $\mathcal{U} \sqcup \mathcal{V}$.

Remark 1.3.3

Since the set $I \sqcup J$ from the previous definition is only a set of pairs of the form "(element,label)", there is no possible duplication, even when $I = J$ (which does not prevent two different elements from being sent to the same open set by $\mathcal{U}$ or $\mathcal{V}$) and the order is not important in a set. Thus, $I \sqcup J = J \sqcup I$, and therefore $\mathcal{U} \sqcup \mathcal{V} = \mathcal{V} \sqcup \mathcal{U}$.

We also define another operation on generalised coverings which will be useful later

Definition 1.3.4 Pull back of generalised converings

Let X, Y topological spaces. And let $f : X \longrightarrow Y$ a continuous map.
Let $\mathcal{U} : I \longrightarrow \mathcal{O}_X$ and $\mathcal{V} : J \longrightarrow \mathcal{O}_Y$ two generalised converings of X and Y.
Then we call *pull back of* $\mathcal{V}$ *in* $\mathcal{U}$ *according* f, that we denote $\mathcal{U}^*_f(\mathcal{V})$, or only $\mathcal{U}^*(\mathcal{V})$, the following covering:

$$\begin{array}{cccc} \mathcal{U}^*_f(\mathcal{V}): & I \times J & \longrightarrow & \mathcal{O}_X \\ & (i,j) & \mapsto & f^{-1}(\mathcal{V}_j) \cap \mathcal{U}_i \end{array}.$$

Remark 1.3.5

We can see that each open set $\mathcal{W}_k$ of $\mathcal{U}^*_f(\mathcal{V})$ is contained in a particular space $\mathcal{U}_i$ with $k = (i,j)$.
This way to "include" the generalised covering $\mathcal{U}^*_f(\mathcal{V})$ in $\mathcal{U}$ will be useful later.

We will now refine the pre-existing topological groupoids to take account of the information provided by a generalised covering of the underlying space.

Definition 1.3.6 Čech groupoïd

If $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ is a topological groupoid and $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in I}$ a generalized covering of $\mathcal{G}^{(0)}$, then the Čech groupoid associated with $\mathcal{G}$ and $\mathcal{U}$, denoted by $\mathcal{G}_\mathcal{U} \rightrightarrows \mathcal{G}_\mathcal{U}^{(0)}$, or only $\mathcal{G}_\mathcal{U}$, is defined by $\bigsqcup_{(\alpha,\beta)\in I^2} \mathcal{G}_{\mathcal{U}_\beta}^{\mathcal{U}_\alpha} \rightrightarrows \bigsqcup_{\alpha \in I} \mathcal{U}_\alpha$, with :

$$\forall\ \alpha, \beta, \gamma \in I,\ \forall\ g \in \mathcal{G}_{\mathcal{U}_\beta}^{\mathcal{U}_\alpha},\ h \in \mathcal{G}_{\mathcal{U}_\gamma}^{\mathcal{U}_\beta},\ \forall\ x \in \mathcal{U}_\alpha,\ \begin{cases} s(\mathcal{U}_\alpha, g, \mathcal{U}_\beta) = (s(g), \mathcal{U}_\beta) \\ t(\mathcal{U}_\alpha, g, \mathcal{U}_\beta) = (t(g), \mathcal{U}_\alpha) \\ (\mathcal{U}_\alpha, g, \mathcal{U}_\beta)(\mathcal{U}_\beta, h, \mathcal{U}_\gamma) = (\mathcal{U}_\alpha, gh, \mathcal{U}_\gamma) \\ i(\mathcal{U}_\alpha, g, \mathcal{U}_\beta) = (\mathcal{U}_\beta, g^{-1}, \mathcal{U}_\alpha) \\ u(x, \mathcal{U}_\alpha) = (\mathcal{U}_\alpha, \mathbb{1}_x, \mathcal{U}_\alpha) \end{cases}.$$

The sets $\bigsqcup_{(\alpha,\beta)\in I^2} \mathcal{G}_{\mathcal{U}_\beta}^{\mathcal{U}_\alpha}$ and $\bigsqcup_{\alpha \in I} \mathcal{U}_\alpha$ are endowed with the disjoint union topology, so that the groupoid $\mathcal{G}_\mathcal{U}$ is a topological groupoid

Each object is assigned a label according the open set they belong to, and each arrow is doubly labeled according to its departure and arrival objects.

Since the open sets can intersect, certain objects and certain arrows are duplicated when there are several open spaces to which they belong.

We now construct the means to define a generalized notion of groupoid morphism.

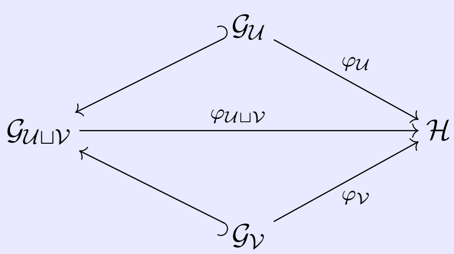
Definition 1.3.7 1-cocycle over groupoids

Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ and $\mathcal{H} \rightrightarrows \mathcal{H}^{(0)}$ be two topological groupoids. We call 1-cocycle from $\mathcal{G}$ to $\mathcal{H}$ a strict morphism from $\mathcal{G}_\mathcal{U}$ to $\mathcal{H}$, where $\mathcal{G}_\mathcal{U}$ denotes the Čech groupoid of $\mathcal{G}$ associated with a covering $\mathcal{U}$ of $\mathcal{G}^{(0)}$.

In the same way that we deal with compatible atlases when we define a manifold structure on a topological space, we are now going to define an equivalence relation on the 1-cocycles of fixed groupoids which translates the concordance of the morphisms.

Definition 1.3.8 Equivalence of 1-cocycles

Let $\varphi_\mathcal{U} : \mathcal{G}_\mathcal{U} \longrightarrow \mathcal{H}$ and $\varphi_\mathcal{V} : \mathcal{G}_\mathcal{V} \longrightarrow \mathcal{H}$ be two cocycles from $\mathcal{G}$ to $\mathcal{H}$ associated with the generalized covering $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{G}^{(0)}$. These two cocycles are said to be equivalent, denoted $\varphi_\mathcal{U} \sim \varphi_\mathcal{V}$, if there exists $\varphi_{\mathcal{U}\sqcup\mathcal{V}} : \mathcal{G}_{\mathcal{U}\sqcup\mathcal{V}} \longrightarrow \mathcal{H}$, a cocycle associated with the generalized covering $\mathcal{U} \sqcup \mathcal{V}$, such that the following diagramm commutes :



where the injectives arrows are the canonical injections of groupoids.

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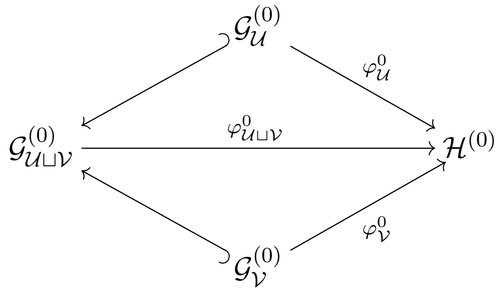
Remark 1.3.9

With the previous notations, it should be noted that, by functoriality, we have the same diagrams for the objects :



Property 1.3.10

The relation thus defined is an equivalence relation on the set of 1-cocycles between two fixed groupoids $\mathcal{G}$ and $\mathcal{H}$.

Proof

Reflexivity :

Let $\varphi_{\mathcal{U}} : \mathcal{G}_{\mathcal{U}} \longrightarrow \mathcal{H}$ be a 1-cocyle associated with a generalized covering $\mathcal{U}$. We denote by $\widetilde{\mathcal{U}}$ and $\widehat{\mathcal{U}}$ the two copies of $\mathcal{U}$ in the disjoint union of $\mathcal{U}$ and $\mathcal{U}$. We then define the 1-cocycle $\varphi_{\widetilde{\mathcal{U}}\sqcup\widehat{\mathcal{U}}}$ associated with $\widetilde{\mathcal{U}}\sqcup\widehat{\mathcal{U}} = \mathcal{U}\sqcup\mathcal{U}$ by :

$$\begin{cases} \forall\ g \in \mathcal{G}_{\widetilde{\mathcal{U}}_j}^{\widetilde{\mathcal{U}}_i},\ \varphi_{\widetilde{\mathcal{U}}\sqcup\widehat{\mathcal{U}}}(\widetilde{\mathcal{U}}_i, g, \widetilde{\mathcal{U}}_j) = \varphi_{\mathcal{U}}(\mathcal{U}_i, g, \mathcal{U}_j) \\ \\ \forall\ g \in \mathcal{G}_{\widehat{\mathcal{U}}_j}^{\widetilde{\mathcal{U}}_i},\ \varphi_{\widetilde{\mathcal{U}}\sqcup\widehat{\mathcal{U}}}(\widetilde{\mathcal{U}}_i, g, \widehat{\mathcal{U}}_j) = \varphi_{\mathcal{U}}(\mathcal{U}_i, g, \mathcal{U}_j) \\ \\ \forall\ g \in \mathcal{G}_{\widetilde{\mathcal{U}}_j}^{\widehat{\mathcal{U}}_i},\ \varphi_{\widetilde{\mathcal{U}}\sqcup\widehat{\mathcal{U}}}(\widehat{\mathcal{U}}_i, g, \widetilde{\mathcal{U}}_j) = \varphi_{\mathcal{U}}(\mathcal{U}_i, g, \mathcal{U}_j) \\ \\ \forall\ g \in \mathcal{G}_{\widehat{\mathcal{U}}_j}^{\widehat{\mathcal{U}}_i},\ \varphi_{\widetilde{\mathcal{U}}\sqcup\widehat{\mathcal{U}}}(\widehat{\mathcal{U}}_i, g, \widehat{\mathcal{U}}_j) = \varphi_{\mathcal{U}}(\mathcal{U}_i, g, \mathcal{U}_j) \end{cases},$$

and also :

$$\begin{cases} \forall\ x \in \widetilde{\mathcal{U}}_i,\ \varphi_{\widetilde{\mathcal{U}}\sqcup\widehat{\mathcal{U}}}^{0}(x, \widetilde{\mathcal{U}}_i) = \varphi_{\mathcal{U}}^{0}(x, \mathcal{U}_i) \\ \\ \forall\ x \in \widehat{\mathcal{U}}_i,\ \varphi_{\widetilde{\mathcal{U}}\sqcup\widehat{\mathcal{U}}}^{0}(x, \widehat{\mathcal{U}}_i) = \varphi_{\mathcal{U}}^{0}(x, \mathcal{U}_i) \end{cases}$$

It is then enough to check the functoriality of $\varphi_{\widetilde{\mathcal{U}}\sqcup\widehat{\mathcal{U}}}$ with respect to the applications $m,\ s,\ t,\ u,\ i$ of $\mathcal{G}_{\widetilde{\mathcal{U}}\sqcup\widehat{\mathcal{U}}}$ in each of the above cases in order to be sure that $\varphi_{\widetilde{\mathcal{U}}\sqcup\widehat{\mathcal{U}}}$ indeed defines a 1-cocycle, we will then have the commutativity of the diagram via the definition of $\varphi_{\widetilde{\mathcal{U}}\sqcup\widehat{\mathcal{U}}}$.

We will only check the functoriality with the source map s in the case of a $g \in \mathcal{G}_{\widehat{\mathcal{U}}_j}^{\widetilde{\mathcal{U}}_i}$:

$$\begin{aligned} s(\varphi_{\widetilde{\mathcal{U}}\sqcup\widehat{\mathcal{U}}}(\widetilde{\mathcal{U}}_i, g, \widehat{\mathcal{U}}_j)) &= s(\varphi_{\mathcal{U}}(\mathcal{U}_i, g, \mathcal{U}_j)) \\ &= \varphi^0_{\mathcal{U}}(s(\mathcal{U}_i, g, \mathcal{U}_j)) \\ &= \varphi^0_{\mathcal{U}}((s(g), \mathcal{U}_j)) \\ &= \varphi^0_{\widetilde{\mathcal{U}}\sqcup\widetilde{\mathcal{U}}}(s(g), \mathcal{U}_j)) \\ &= \varphi^0_{\widetilde{\mathcal{U}}\sqcup\widetilde{\mathcal{U}}}(s(\widetilde{\mathcal{U}}_i, g, \widehat{U}_j)). \end{aligned}$$

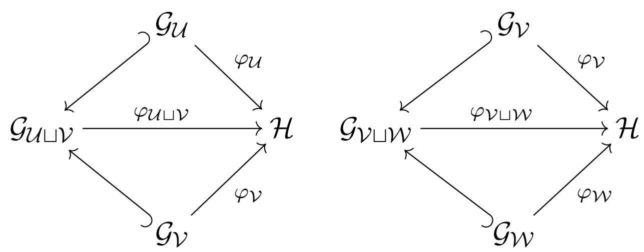
In each case, the functoriality of $\varphi_{\widetilde{\mathcal{U}}\sqcup\widehat{\mathcal{U}}}$ is a consequence of that of $\varphi_{\mathcal{U}}$.

Symmetry :

Since the sum of two generalized coverings is a commutative operation, the definition of the equivalence relation is symmetric, which concludes.

Transitivity :

Let $\varphi_{\mathcal{U}} : \mathcal{G}_{\mathcal{U}} \longrightarrow \mathcal{H}$, $\varphi_{\mathcal{V}} : \mathcal{G}_{\mathcal{V}} \longrightarrow \mathcal{H}$, and $\varphi_{\mathcal{U}} : \mathcal{G}_{\mathcal{W}} \longrightarrow \mathcal{H}$ be 1-cocycles associated respectively to $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$. We suppose that $\varphi_{\mathcal{U}} \sim \varphi_{\mathcal{V}}$ et $\varphi_{\mathcal{V}} \sim \varphi_{\mathcal{W}}$. Thus, we have $\varphi_{\mathcal{U}\sqcup\mathcal{V}} : \mathcal{G}_{\mathcal{U}\sqcup\mathcal{V}} \longrightarrow \mathcal{H}$ et $\varphi_{\mathcal{V}\sqcup\mathcal{W}} : \mathcal{G}_{\mathcal{V}\sqcup\mathcal{W}} \longrightarrow \mathcal{H}$ two 1-cocycles associated respectively to $\mathcal{U} \sqcup \mathcal{V}$ and $\mathcal{V} \sqcup \mathcal{W}$, such that the diagramms



commutes.

Let's construct $\varphi_{\mathcal{U}\sqcup\mathcal{W}} : \mathcal{G}_{\mathcal{U}\sqcup\mathcal{W}} \longrightarrow \mathcal{H}$.

For $x \in \mathcal{U}_j$, we have $\varphi^0_{\mathcal{U}\sqcup\mathcal{W}}(x, \mathcal{U}_j) = \varphi^0_{\mathcal{U}}(x, \mathcal{U}_j)$ and for $x \in \mathcal{W}_j$, $\varphi^0_{\mathcal{U}\sqcup\mathcal{W}}(x, \mathcal{W}_j) = \varphi^0_{\mathcal{U}}(x, \mathcal{W}_j)$.
If $g \in \mathcal{G}^{\mathcal{U}_i}_{\mathcal{U}_j}$, then $\varphi_{\mathcal{U}\sqcup\mathcal{W}}(\mathcal{U}_i, q, \mathcal{U}_j) = \varphi_{\mathcal{U}}(\mathcal{U}_i, g, \mathcal{U}_j)$ and, if $g \in \mathcal{G}^{\mathcal{W}_i}_{\mathcal{W}_j}$, then $\varphi_{\mathcal{U}\sqcup\mathcal{W}}(\mathcal{W}_i, g, \mathcal{W}_j) = \varphi_{\mathcal{W}}(\mathcal{W}_i, g, \mathcal{W}_j)$.

Let $g \in \mathcal{G}^{\mathcal{W}_i}_{\mathcal{U}_j}$. We search how to define $\varphi_{\mathcal{U}\sqcup\mathcal{W}}(\mathcal{W}_i, g, \mathcal{U}_j)$. The idea is that, to go from $\mathcal{U}$ to $\mathcal{W}$, we can go through $\mathcal{V}$ in a canonical way.

Because $\mathcal{V} = \{\mathcal{V}_k\}_{k\in K}$ is a generalized covering of $\mathcal{G}^{(0)}$. We have $k, l \in K$, such that $s(g) \in \mathcal{V}_k$ and $t(g) \in \mathcal{V}_l$. Thus, we have the following decomposition :

$$(\mathcal{W}_i, g, \mathcal{U}_j) = (\mathcal{W}_i, \mathbb{1}_{t(g)}, \mathcal{V}_l)(\mathcal{V}_l, g, \mathcal{V}_k)(\mathcal{V}_k, \mathbb{1}_{s(g)}, \mathcal{U}_j),$$

which leads us to set $\varphi_{\mathcal{V}\sqcup\mathcal{W}}(\mathcal{W}_i, \mathbb{1}_{t(g)}, \mathcal{V}_l)\varphi_{\mathcal{V}}(\mathcal{V}_l, g, \mathcal{V}_k)\varphi_{\mathcal{U}\sqcup\mathcal{V}}(\mathcal{V}_k, \mathbb{1}_{s(g)}, \mathcal{U}_j)$, where the different terms are all composable thanks to the functoriality of the applications $\varphi_{\mathcal{V}\sqcup\mathcal{W}}$, $\varphi_{\mathcal{V}}$, $\varphi_{\mathcal{U}\sqcup\mathcal{V}}$.

Let us verify that this quantity is independent of the choice of the open sets $\mathcal{V}_k$ and $\mathcal{V}_l$. If $s(g) \in \mathcal{V}_k \cap \mathcal{V}_{k'}$ and $t(g) \in \mathcal{V}_l \cap \mathcal{V}_{l'}$, then :

$$\begin{aligned}
&\varphi_{\mathcal{V}\sqcup\mathcal{W}}(\mathcal{W}_i, \mathbb{1}_{t(g)}, \mathcal{V}_l)\ \ \varphi_{\mathcal{V}}(\mathcal{V}_l, g, \mathcal{V}_k)\ \ \varphi_{\mathcal{U}\sqcup\mathcal{V}}(\mathcal{V}_k, \mathbb{1}_{s(g)}, \mathcal{U}_j)\\
&= \varphi_{\mathcal{V}\sqcup\mathcal{W}}(\mathcal{W}_i, \mathbb{1}_{t(g)}, \mathcal{V}_l)\ \ \varphi_{\mathcal{V}}((\mathcal{V}_l, \mathbb{1}_{t(g)}, \mathcal{V}_{l'})(\mathcal{V}_{l'}, g, \mathcal{V}_{k'})(\mathcal{V}_{k'}, \mathbb{1}_{s(g)}, \mathcal{V}_k))\ \ \varphi_{\mathcal{U}\sqcup\mathcal{V}}(\mathcal{V}_k, \mathbb{1}_{s(g)}, \mathcal{U}_j)\\
&= \varphi_{\mathcal{V}\sqcup\mathcal{W}}(\mathcal{W}_i, \mathbb{1}_{t(g)}, \mathcal{V}_l)\ \ \varphi_{\mathcal{V}\sqcup\mathcal{W}}(\mathcal{V}_l, \mathbb{1}_{t(g)}, \mathcal{V}_{l'})\ \ \varphi_{\mathcal{V}}(\mathcal{V}_{l'}, g, \mathcal{V}_{k'})\ \ \varphi_{\mathcal{U}\sqcup\mathcal{V}}(\mathcal{V}_{k'}, \mathbb{1}_{s(g)}, \mathcal{V}_k))\ \ \varphi_{\mathcal{U}\sqcup\mathcal{V}}(\mathcal{V}_k, \mathbb{1}_{s(g)}, \mathcal{U}_j)\\
&= \varphi_{\mathcal{V}\sqcup\mathcal{W}}(\mathcal{W}_i, \mathbb{1}_{t(g)}, \mathcal{V}_{l'})\ \ \varphi_{\mathcal{V}}(\mathcal{V}_{l'}, g, \mathcal{V}_{k'})\ \ \varphi_{\mathcal{U}\sqcup\mathcal{V}}(\mathcal{V}_{k'}, \mathbb{1}_{s(g)}, \mathcal{U}_j).
\end{aligned}$$

We can therefore state unambiguously :

$$\varphi_{\mathcal{U}\sqcup\mathcal{W}}(\mathcal{W}_i, g, \mathcal{U}_j) = \varphi_{\mathcal{V}\sqcup\mathcal{W}}(\mathcal{W}_i, \mathbb{1}_{t(g)}, \mathcal{V}_l)\ \ \varphi_{\mathcal{V}}(\mathcal{V}_l, g, \mathcal{V}_k)\ \ \varphi_{\mathcal{U}\sqcup\mathcal{V}}(\mathcal{V}_k, \mathbb{1}_{s(g)}, \mathcal{U}_j),$$

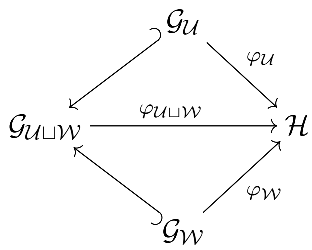
where $s(g) \in \mathcal{V}_k$ and $t(g) \in \mathcal{V}_l$. Similarly, for $g \in \mathcal{G}_{\mathcal{W}_j}^{\mathcal{U}_i}$, we set :

$$\varphi_{\mathcal{U}\sqcup\mathcal{W}}(\mathcal{U}_i, g, \mathcal{W}_j) = \varphi_{\mathcal{U}\sqcup\mathcal{V}}(\mathcal{U}_i, \mathbb{1}_{t(g)}, \mathcal{V}_l)\ \ \varphi_{\mathcal{V}}(\mathcal{V}_l, g, \mathcal{V}_k)\ \ \varphi_{\mathcal{V}\sqcup\mathcal{W}}(\mathcal{V}_k, \mathbb{1}_{s(g)}, \mathcal{W}_j),$$

without ambiguity. It therefore remains to verify that, thus defined, $\varphi_{\mathcal{U}\sqcup\mathcal{W}}$ is a 1-cocycle. Let's check the functionality with respect to the source map in one case, the other checks will be done on the same model. Consider the case where $g \in \mathcal{G}_{\mathcal{U}_j}^{\mathcal{W}_i}$. Then , we have :

$$\begin{aligned}
&s(\varphi_{\mathcal{U}\sqcup\mathcal{W}}(\mathcal{W}_i, g, \mathcal{U}_j))\\
&= s(\varphi_{\mathcal{V}\sqcup\mathcal{W}}(\mathcal{W}_i, \mathbb{1}_{t(g)}, \mathcal{V}_l)\ \ \varphi_{\mathcal{V}}(\mathcal{V}_l, g, \mathcal{V}_k)\ \ \varphi_{\mathcal{U}\sqcup\mathcal{V}}(\mathcal{V}_k, \mathbb{1}_{s(g)}, \mathcal{U}_j))\\
&= s(\varphi_{\mathcal{U}\sqcup\mathcal{V}}(\mathcal{V}_k, \mathbb{1}_{s(g)}, \mathcal{U}_j))\\
&= \varphi^0_{\mathcal{U}\sqcup\mathcal{V}}(s(g), \mathcal{U}_j)\\
&= \varphi^0_{\mathcal{U}}(s(g), \mathcal{U}_j)\\
&= \varphi^0_{\mathcal{U}\sqcup\mathcal{W}}(s(g), \mathcal{U}_j)\\
&= \varphi^0_{\mathcal{U}\sqcup\mathcal{W}}(s(\mathcal{W}_i, g, \mathcal{U}_j)).
\end{aligned}$$

Thus, $\varphi_{\mathcal{U}\sqcup\mathcal{W}} : \mathcal{G}_{\mathcal{U}\sqcup\mathcal{W}} \longrightarrow \mathcal{H}$ is a cocycle and by construction, the diagram



commutes. So we have $\varphi_{\mathcal{U}} \sim \varphi_{\mathcal{W}}$.

We therefore have an equivalence relation.

Definition 1.3.11 Hilsum-Skandalis morphism

If $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ and $\mathcal{H} \rightrightarrows \mathcal{H}^{(0)}$ are two topological groupoids, we define the Hilsum-Skandalis morphism from $\mathcal{G}$ to $\mathcal{H}$ as an equivalence class of 1-cocycle for the previous equivalence relation.

For a 1-cocycle $\varphi_{\mathcal{U}} : \mathcal{G}_{\mathcal{U}} \longrightarrow \mathcal{H}$, we will denote its class $[\varphi_{\mathcal{U}}] : \mathcal{G} \dashrightarrow \mathcal{H}$, or only $\varphi_{\mathcal{U}} : \mathcal{G} \dashrightarrow \mathcal{H}$.

The class of Hilsum-Skandalis morphisms will be denoted by $Grpd_{HS}(\mathcal{G},\mathcal{H})$.

Property 1.3.12

We have a canonical map from the set of strict morphisms in the set of Hilsum-Skandalis morphisms : $Grpd(\mathcal{G},\mathcal{H}) \to Grpd_{HS}(\mathcal{G},\mathcal{H})$.

For $\varphi : \mathcal{G} \longrightarrow \mathcal{H}$ a strict morphism from the groupoïd $\mathcal{G}$ to $\mathcal{H}$, we will denote its image in $Grpd_{HS}(\mathcal{G},\mathcal{H})$ with $[\varphi]$, or only φ again, if there is no risk of confusion.

Proof

Let $\mathcal{G} \xrightarrow{\varphi} \mathcal{H}$ a strict groupoïd morphism. We set $\mathcal{U} = \{\mathcal{G}^{(0)}\}$ the trivial generalised covering of $\mathcal{G}^{(0)}$.
Then we set the cocycle $\varphi_{\mathcal{U}}$ defined by:

$$\forall g \in \mathcal{G},\ \varphi_{\mathcal{U}}(\mathcal{G}^{(0)}, g, \mathcal{G}^{(0)}) = \varphi(g) \qquad ; \qquad \forall x \in \mathcal{G}^{(0)},\ \varphi^0_{\mathcal{U}}(x, \mathcal{G}^{(0)}) = \varphi^0(x).$$

Then we can immediately check that we have a 1-cocycle from $\mathcal{G}$ to $\mathcal{H}$:

$$\begin{array}{ccc} \mathcal{G}_{\mathcal{U}} & \xrightarrow{\varphi_{\mathcal{U}}} & \mathcal{H} \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathcal{G}^{(0)}_{\mathcal{U}} & \xrightarrow[\varphi^0_{\mathcal{U}}]{} & \mathcal{H}^{(0)}_{\mathcal{U}} \end{array}$$

.

For this reason, we will sometimes speak of generalized morphisms instead of Hilsum-Skandalis morphisms.
Now we have a new notion of isomorphisms, then it is natural to try to compose them. At first we will see what a "composition" of 1-cocycle looks like, and then we will check that it is well defined according the 1-cocycle equivalence.

Let $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{K}$ three topological groupoïds. Let $\varphi_{\mathcal{U}}$, $\varphi_{\mathcal{V}}$ two 1-cocyles from $\mathcal{G}$ to $\mathcal{H}$ and from $\mathcal{H}$ to $\mathcal{K}$ associated to the generalised converings $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{G}^{(0)}$ and $\mathcal{H}^{(0)}$. In total we have:

$$\begin{array}{ccc} \mathcal{G}_{\mathcal{U}} & \xrightarrow{\varphi_{\mathcal{U}}} & \mathcal{H} \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathcal{G}^{(0)}_{\mathcal{U}} & \xrightarrow[\varphi^0_{\mathcal{U}}]{} & \mathcal{H}^{(0)}_{\mathcal{U}} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{H}_{\mathcal{V}} & \xrightarrow{\varphi_{\mathcal{V}}} & \mathcal{K} \\ \downarrow\downarrow & & \downarrow\downarrow \\ \mathcal{H}^{(0)}_{\mathcal{V}} & \xrightarrow[\varphi^0_{\mathcal{V}}]{} & \mathcal{K}^{(0)} \end{array}$$

Then now, to define a 1-cocycle from $\mathcal{G}$ to $\mathcal{K}$ which could correspond to a composition, let $\mathcal{W} = \{\mathcal{W}_{i,j}\}_{i,j}$ be the generalised covering $\mathcal{U}^*_{\varphi^0_{\mathcal{U}}}(\mathcal{V})$ which is the pull back of $\mathcal{V}$ in $\mathcal{U}$ according $\varphi^0_{\mathcal{U}}$. We recall that we have the privileged inclusions $\mathcal{W}_{i,j} \subset \mathcal{U}_i$ and $\varphi^0_{\mathcal{U}}(\mathcal{W}_{i,j}) \subset \mathcal{V}_j$. Then we define $\varphi_{\mathcal{W}}$ by:

$$\forall g \in \mathcal{G}^{\mathcal{W}_{i_1,j_1}}_{\mathcal{W}_{i_2,j_2}},\ \ \varphi_{\mathcal{W}}(\mathcal{W}_{i_1,j_1}, g, \mathcal{W}_{i_2,j_2}) = \varphi_{\mathcal{V}}(\mathcal{V}_{j_1}, \varphi_{\mathcal{U}}(\mathcal{U}_{i_1}, g, \mathcal{U}_{i_2}), \mathcal{V}_{j_2})$$
$$\forall x \in \mathcal{W}_{i,j},\ \ \varphi^0_{\mathcal{W}}(x, \mathcal{W}_{i,j}) = \varphi^0_{\mathcal{V}}(\varphi^0_{\mathcal{U}}(x, \mathcal{U}_i), \mathcal{V}_j)$$

We have to check that it defines a 1-cocycle, once again we will only show the compatibility with the source map, the others compatibilities are proved the same way. For g in $\mathcal{G}_{\mathcal{W}_{i_2,j_2}}^{\mathcal{W}_{i_1,j_1}}$,

$$\begin{aligned} s(\varphi_{\mathcal{W}}(\mathcal{W}_{i_1,j_1},g,\mathcal{W}_{i_2,j_2})) &= s(\varphi_{\mathcal{V}}(\mathcal{V}_{j_1},\varphi_{\mathcal{U}}(\mathcal{U}_{i_1},g,\mathcal{U}_{i_2}),\mathcal{V}_{j_2}))\\ &= \varphi^0_{\mathcal{V}}(s(\mathcal{V}_{j_1},\varphi_{\mathcal{U}}(\mathcal{U}_{i_1},g,\mathcal{U}_{i_2}),\mathcal{V}_{j_2}))\\ &= \varphi^0_{\mathcal{V}}(s(\varphi_{\mathcal{U}}(\mathcal{U}_{i_1},g,\mathcal{U}_{i_2})),\mathcal{V}_{j_2})\\ &= \varphi^0_{\mathcal{V}}(\varphi^0_{\mathcal{U}}(s(g),\mathcal{U}_{i_2}),\mathcal{V}_{j_2})\\ &= \varphi^0_{\mathcal{W}}(s(g),\mathcal{W}_{i_2,j_2})\\ &= \varphi^0_{\mathcal{W}}(s(\mathcal{W}_{i_1,j_1},g,\mathcal{W}_{i_2,j_2})) \end{aligned}$$

Then we have a 1-cocycle.

If we denote $\varphi_{\mathcal{U}^*(\mathcal{V})}$ the 1-cocycle we just built from $\varphi_{\mathcal{U}}$ and $\varphi_{\mathcal{V}}$, we have to check that this construction is coherent with the quotient, namely:

Property 1.3.13

Let $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$ three topological groupoïds. Using the previous notations, the map:

$$\begin{array}{ccc} G_{HS}(\mathcal{G},\mathcal{H})\times G_{HS}(\mathcal{H},\mathcal{K}) & \longrightarrow & G_{HS}(\mathcal{G},\mathcal{K})\\ ([\varphi_{\mathcal{U}}],[\varphi_{\mathcal{V}}]) & \mapsto & \left[\varphi_{\mathcal{U}^*(\mathcal{V})}\right] \end{array}$$

is well defined.

Proof

Let $\varphi_{\mathcal{U}}$, $\varphi_{\widetilde{\mathcal{U}}}$ two 1-cocycles from $\mathcal{G}$ to $\mathcal{H}$ associated to the generalised coverings $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ of $\mathcal{G}^{(0)}$.
Let $\varphi_{\mathcal{V}}$, $\varphi_{\widetilde{\mathcal{V}}}$ two 1-cocycles from $\mathcal{H}$ to $\mathcal{K}$ associated to the generalised coverings $\mathcal{V}$ and $\widetilde{\mathcal{V}}$ of $\mathcal{H}^{(0)}$.
We suppose that $\varphi_{\mathcal{U}}\sim\varphi_{\widetilde{\mathcal{U}}}$ and $\varphi_{\mathcal{V}}\sim\varphi_{\widetilde{\mathcal{V}}}$. Then we get $\varphi_{\mathcal{U}\sqcup\widetilde{\mathcal{U}}}:\mathcal{G}_{\mathcal{U}\sqcup\widetilde{\mathcal{U}}}\longrightarrow\mathcal{H}$ and $\varphi_{\mathcal{V}\sqcup\widetilde{\mathcal{V}}}:\mathcal{H}_{\mathcal{V}\sqcup\widetilde{\mathcal{V}}}\longrightarrow\mathcal{K}$ such that the 1-cocycle equivalence diagrams commute.
Let's show that $\varphi_{\mathcal{U}^*(\mathcal{V})}\sim\varphi_{\widetilde{\mathcal{U}}^*(\widetilde{\mathcal{V}})}$:
We denote $\mathcal{W}=\mathcal{U}^*(\mathcal{V})$ and $\widetilde{\mathcal{W}}=\widetilde{\mathcal{U}}^*(\widetilde{\mathcal{V}})$. Then we set $\varphi_{\mathcal{W}\sqcup\widetilde{\mathcal{W}}}$ to be:

$$\begin{cases} \forall\ g\in\mathcal{G}_{\mathcal{W}_{i_2,j_2}}^{\mathcal{W}_{i_1,j_1}}, & \varphi_{\mathcal{W}\sqcup\widetilde{\mathcal{W}}}(\mathcal{W}_{i_1,j_1},g,\mathcal{W}_{i_2,j_2})=\varphi_{\mathcal{V}\sqcup\widetilde{\mathcal{V}}}(\mathcal{V}_{j_1},\ \varphi_{\mathcal{U}\sqcup\widetilde{\mathcal{U}}}(\mathcal{U}_{i_1},g,\mathcal{U}_{i_2}),\ \mathcal{V}_{j_2})\\ \forall\ g\in\mathcal{G}_{\widetilde{\mathcal{W}}_{i_2,j_2}}^{\widetilde{\mathcal{W}}_{i_1,j_1}}, & \varphi_{\mathcal{W}\sqcup\widetilde{\mathcal{W}}}(\widetilde{\mathcal{W}}_{i_1,j_1},g,\widetilde{\mathcal{W}}_{i_2,j_2})=\varphi_{\mathcal{V}\sqcup\widetilde{\mathcal{V}}}(\widetilde{\mathcal{V}}_{j_1},\ \varphi_{\mathcal{U}\sqcup\widetilde{\mathcal{U}}}(\widetilde{\mathcal{U}}_{i_1},g,\widetilde{\mathcal{U}}_{i_2}),\ \widetilde{\mathcal{V}}_{j_2})\\ \forall\ g\in\mathcal{G}_{\mathcal{W}_{i_2,j_2}}^{\widetilde{\mathcal{W}}_{i_1,j_1}}, & \varphi_{\mathcal{W}\sqcup\widetilde{\mathcal{W}}}(\widetilde{\mathcal{W}}_{i_1,j_1},g,\mathcal{W}_{i_2,j_2})=\varphi_{\mathcal{V}\sqcup\widetilde{\mathcal{V}}}(\widetilde{\mathcal{V}}_{j_1},\ \varphi_{\mathcal{U}\sqcup\widetilde{\mathcal{U}}}(\widetilde{\mathcal{U}}_{i_1},g,\mathcal{U}_{i_2}),\ \mathcal{V}_{j_2})\\ \forall\ g\in\mathcal{G}_{\widetilde{\mathcal{W}}_{i_2,j_2}}^{\mathcal{W}_{i_1,j_1}}, & \varphi_{\mathcal{W}\sqcup\widetilde{\mathcal{W}}}(\mathcal{W}_{i_1,j_1},g,\widetilde{\mathcal{W}}_{i_2,j_2})=\varphi_{\mathcal{V}\sqcup\widetilde{\mathcal{V}}}(\mathcal{V}_{j_1},\ \varphi_{\mathcal{U}\sqcup\widetilde{\mathcal{U}}}(\mathcal{U}_{i_1},g,\widetilde{\mathcal{U}}_{i_2}),\ \widetilde{\mathcal{V}}_{j_2}) \end{cases},$$

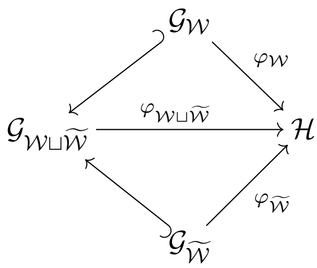
and

$$\begin{cases} \forall x\in\mathcal{W}_{i,j},\ \varphi^0_{\mathcal{W}\sqcup\widetilde{\mathcal{W}}}(x,\mathcal{W}_{i,j})=\varphi^0_{\mathcal{V}\sqcup\widetilde{\mathcal{V}}}(\varphi^0_{\mathcal{U}\sqcup\widetilde{\mathcal{U}}}(x,\mathcal{U}_i),\mathcal{V}_j)\\ \forall x\in\widetilde{\mathcal{W}}_{i,j},\ \varphi^0_{\mathcal{W}\sqcup\widetilde{\mathcal{W}}}(x,\widetilde{\mathcal{W}}_{i,j})=\varphi^0_{\mathcal{V}\sqcup\widetilde{\mathcal{V}}}(\varphi^0_{\mathcal{U}\sqcup\widetilde{\mathcal{U}}}(x,\widetilde{\mathcal{U}}_i),\widetilde{\mathcal{V}}_j) \end{cases}$$

To check this define a 1-cocycle, once again we check it only with the source map, and only in one case : $g \in \mathcal{G}_{\widetilde{\mathcal{W}}_{i_2,j_2}}^{\mathcal{W}_{i_1,j_1}}$.

$$\begin{aligned} s(\varphi_{\mathcal{W}\sqcup\widetilde{\mathcal{W}}}(\mathcal{W}_{i_1,j_1}, g, \widetilde{\mathcal{W}}_{i_2,j_2})) &= s(\varphi_{\mathcal{V}\sqcup\widetilde{\mathcal{V}}}(\mathcal{V}_{j_1}, \varphi_{\mathcal{U}\sqcup\widetilde{\mathcal{U}}}(\mathcal{U}_{i_1}, g, \widetilde{\mathcal{U}}_{i_2}), \widetilde{\mathcal{V}}_{j_2})) \\ &= \varphi^0_{\mathcal{V}\sqcup\widetilde{\mathcal{V}}}(s(\mathcal{V}_{j_1}, \varphi_{\mathcal{U}\sqcup\widetilde{\mathcal{U}}}(\mathcal{U}_{i_1}, g, \widetilde{\mathcal{U}}_{i_2}), \widetilde{\mathcal{V}}_{j_2})) \\ &= \varphi^0_{\mathcal{V}\sqcup\widetilde{\mathcal{V}}}(s(\varphi_{\mathcal{U}\sqcup\widetilde{\mathcal{U}}}(\mathcal{U}_{i_1}, g, \widetilde{\mathcal{U}}_{i_2})), \widetilde{\mathcal{V}}_{j_2}) \\ &= \varphi^0_{\mathcal{V}\sqcup\widetilde{\mathcal{V}}}(\varphi^0_{\mathcal{U}\sqcup\widetilde{\mathcal{U}}}(s(g), \widetilde{\mathcal{U}}_{i_2}), \widetilde{\mathcal{V}}_{j_2}) \\ &= \varphi^0_{\mathcal{W}\sqcup\widetilde{\mathcal{W}}}(s(g), \widetilde{\mathcal{W}}_{i_2,j_2}) \\ &= \varphi^0_{\mathcal{W}\sqcup\widetilde{\mathcal{W}}}(s(\mathcal{W}_{i_1,j_1}, g, \widetilde{\mathcal{W}}_{i_2,j_2})). \end{aligned}$$

Then we indeed have a 1-cocycle, and by definition we indeed have the commutativity of the 1-cocycle equivalence diagram:



Then $\varphi_{\mathcal{U}^*(\mathcal{V})} = \varphi_{\mathcal{W}} \sim \varphi_{\widetilde{\mathcal{W}}} = \varphi_{\widetilde{\mathcal{U}}^*(\widetilde{\mathcal{V}})}$. The map is well defined.

Definition 1.3.14 Hilsum-Skandalis morphisms composition

With the previous notations, the class $\left[\varphi_{\mathcal{U}^*(\mathcal{V})}\right]$ will be denoted $[\varphi_{\mathcal{V}}] \circ [\varphi_{\mathcal{U}}]$, and called *Composition* of the Hilsum-Skandalis morphisms $[\varphi_{\mathcal{V}}]$ and $[\varphi_{\mathcal{U}}]$. We will also refer to the 1-cocyle $\varphi_{\mathcal{U}^*(\mathcal{V})}$ as *pull back 1-cocycle of* $\varphi_{\mathcal{V}}$ *by* $\varphi_{\mathcal{U}}$.

Then we have a way to compose Hilsum-Skandalis morphisms:

$$\begin{array}{rccc} \circ : & G_{HS}(\mathcal{G},\mathcal{H}) \times G_{HS}(\mathcal{H},\mathcal{K}) & \longrightarrow & G_{HS}(\mathcal{G},\mathcal{K}) \\ & ([\varphi_{\mathcal{U}}],[\varphi_{\mathcal{V}}]) & \mapsto & [\varphi_{\mathcal{V}}] \circ [\varphi_{\mathcal{U}}] \end{array}$$

Property 1.3.15 Associativity of the composition - Admitted

Let $\mathcal{G}$, $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ four topologcial groupoïds.
Let $([\varphi_{\mathcal{U}}],[\varphi_{\mathcal{V}}],[\varphi_{\mathcal{W}}]) \in G_{HS}(\mathcal{G},\mathcal{H}) \times G_{HS}(\mathcal{H},\mathcal{K}) \times G_{HS}(\mathcal{K},\mathcal{L})$,
then we have $\big([\varphi_{\mathcal{W}}] \circ [\varphi_{\mathcal{V}}]\big) \circ [\varphi_{\mathcal{U}}] = [\varphi_{\mathcal{W}}] \circ \big([\varphi_{\mathcal{V}}] \circ [\varphi_{\mathcal{U}}]\big)$.

Property 1.3.16 Neutral element

Let $\mathcal{G}$, $\mathcal{H}$ two topological goupoïds. If we denote $Id_{\mathcal{G}}$ and $Id_{\mathcal{H}}$ the strict identity morphism of $\mathcal{G}$ and $\mathcal{H}$, then :

$$\forall\, ([\varphi_{\mathcal{U}}] : \mathcal{G} \dashrightarrow \mathcal{H}) \in G_{HS}(\mathcal{G},\mathcal{H}),\ [\varphi_{\mathcal{U}}] \circ [Id_{\mathcal{G}}] = [\varphi_{\mathcal{U}}] = [Id_{\mathcal{H}}] \circ [\varphi_{\mathcal{U}}].$$

Proof

We will only prove the first equality, the second one is obtained by the same process.
Let's compute the pull-back 1-cocycle of $\varphi_{\mathcal{U}}$ by $Id_{\mathcal{G}}$, we call it f, then:
$\mathcal{G}_{\{\mathcal{G}^{(0)}\}} \xrightarrow{Id_{\mathcal{G}}} \mathcal{G} \qquad \mathcal{G}_{\mathcal{U}} \xrightarrow{\varphi_{\mathcal{U}}} \mathcal{H}$.

$\{\mathcal{G}^{(0)}\}^*(\mathcal{U})_{i,j} = Id_{\mathcal{G}}^{-1}(\mathcal{U}_j) \cap \mathcal{G}^{(0)} = \{((x, \mathcal{G}^{(0)}), \mathcal{U}_j) : x \in \mathcal{U}_j\}$.
$f^0((x, \mathcal{G}^{(0)}), \mathcal{U}_j) = \varphi_{\mathcal{U}}(Id_{\mathcal{G}}(x, \mathcal{G}^{(0)}), \mathcal{U}_j) = \varphi_{\mathcal{U}}(x, \mathcal{U}_j)$, and
$f(\mathcal{U}_i, (\mathcal{G}^{(0)}, g, \mathcal{G}^{(0)}), \mathcal{U}_j) = \varphi_{\mathcal{U}}(\mathcal{U}_i, Id_{\mathcal{G}}(\mathcal{G}^{(0)}, g, \mathcal{G}^{(0)}), \mathcal{U}_j) = \varphi_{\mathcal{U}}(\mathcal{U}_i, g, \mathcal{U}_j)$.
Then we can easily check that this 1-cocycle f is equivalent to $\varphi_{\mathcal{U}}$.
Eventually $[\varphi_{\mathcal{U}}] \circ [Id_{\mathcal{G}}] = [f] = [\varphi_{\mathcal{U}}]$.

Definition 1.3.17 Hilsum-Skandalis category

We denote $Grpd_{HS}$ the category whose objects are the topological groupoïds, and whose the morphisms from $\mathcal{G}$ to $\mathcal{H}$ are the elements of $Grpd_{HS}(\mathcal{G}, \mathcal{H})$. The composition of morphisms is the composition of Hilsum-Skandalis morphisms defined above.

Definition 1.3.18 Morita equivalence between groupoïds

Two topological groupoïds $\mathcal{G}$ and $\mathcal{H}$ will said to be *Morita equivalent* if there exist two Hilsum-Skandalis morphisms $[\varphi] : \mathcal{G} \dashrightarrow \mathcal{H}$ and $[\psi] : \mathcal{H} \dashrightarrow \mathcal{G}$ such that:

$$[\psi] \circ [\varphi] = [Id_{\mathcal{G}}] \text{ and } [\varphi] \circ [\psi] = [Id_{\mathcal{H}}].$$

In this case we will denote $\mathcal{G} \overset{\mathcal{M}}{\sim} \mathcal{H}$.
The relation it defines is an equivalence relation.

Example 1.3.19 Čech groupoïds

Every topological groupoïd is Morita equivalent to its Čech groupoïds.

Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ and $\mathcal{G}_{\mathcal{U}} \rightrightarrows \mathcal{G}_{\mathcal{U}}^{(0)}$ a topological groupoïd and one of its Čech groupoïd associated to a generalised covering $\mathcal{U}$ of $\mathcal{G}^{(0)}$.
For this proof, the Čech groupoïd $\mathcal{G}_{\mathcal{U}} \rightrightarrows \mathcal{G}_{\mathcal{U}}^{(0)}$ will be denoted $\mathcal{H} \rightrightarrows \mathcal{H}^{(0)}$ to avoid confusions.

Let $\mathcal{H} \overset{[\varphi]}{\dashrightarrow} \mathcal{G}$ and $\mathcal{H} \overset{[\psi_{\mathcal{U}}]}{\dashrightarrow} \mathcal{G}$ obtained by the 1-cocycles:

$$\varphi : \begin{array}{ccc} \mathcal{H}_{\mathcal{G}_{\mathcal{U}}^{(0)}} & \longrightarrow & \mathcal{G} \\ (\mathcal{G}_{\mathcal{U}}^{(0)}, (\mathcal{U}_i, g, \mathcal{G}_j), \mathcal{G}_{\mathcal{U}}^{(0)}) & \mapsto & g \\ ((x, \mathcal{U}_i), \mathcal{G}_{\mathcal{U}}^{(0)}) & \mapsto & x \end{array} \quad ; \quad \psi_{\mathcal{U}} : \begin{array}{ccc} \mathcal{G}_{\mathcal{U}} = \mathcal{H} & \longrightarrow & \mathcal{H} \\ (\mathcal{U}_i, g, \mathcal{U}_j) & \mapsto & (\mathcal{U}_i, g, \mathcal{U}_j) \\ (x, \mathcal{U}_i) & \mapsto & (x, \mathcal{U}_i) \end{array}$$

Now we will show that $[\varphi] \circ [\psi_{\mathcal{U}}] = [Id_{\mathcal{G}}]$ and $[\psi_{\mathcal{U}}] \circ [\varphi] = [Id_{\mathcal{H}}]$. Let's compute these compositions as computing the pull-back morphism that we used to define the composition of Hilsum-Skandalis morphisms.

Computation of $[\varphi]\circ[\psi_{\mathcal{U}}]$:

Let's compute the pull-back 1-cocycle : $\quad \mathcal{G}_{\mathcal{U}} \xrightarrow{\psi_{\mathcal{U}}} \mathcal{H} \quad \mathcal{H}_{\{\mathcal{G}_{\mathcal{U}}^{(0)}\}} \xrightarrow{\varphi} \mathcal{G}$.

The pull-back generalised covering is defined as $\mathcal{U}^*_{\psi_{\mathcal{U}}}(\{\mathcal{G}_{\mathcal{U}}^{(0)}\})_{i,*} = (\psi^0_{\mathcal{U}})^{-1}(\mathcal{G}_{\mathcal{U}}^{(0)}) \cap \mathcal{U}_i = \mathcal{U}_i$.
Let $f : \mathcal{G}_{\mathcal{U}^*_{\psi_{\mathcal{U}}}(\{\mathcal{G}_{\mathcal{U}}^{(0)}\})} = \mathcal{G}_{\mathcal{U}} \longrightarrow \mathcal{G}$ the pull-back 1-cocycle, it is defined by :
$f^0(x,\mathcal{U}_j) = \varphi^0(\psi^0_{\mathcal{U}}(x,\mathcal{U}_j),\mathcal{G}_{\mathcal{U}}^{(0)}) = x$ and $f(\mathcal{U}_i,g,\mathcal{U}_j) = \varphi(\mathcal{G}_{\mathcal{U}}^{(0))},\psi_{\mathcal{U}}(\mathcal{U}_i,g,\mathcal{U}_j),\mathcal{G}_{\mathcal{U}}^{(0)}) = g$.
Then we check immediately that f is equivalent to the canonical 1-cocycle induced by $Id_{\mathcal{G}}$.
Then $[Id_{\mathcal{G}}] = [f] = [\varphi]\circ[\psi_{\mathcal{U}}]$

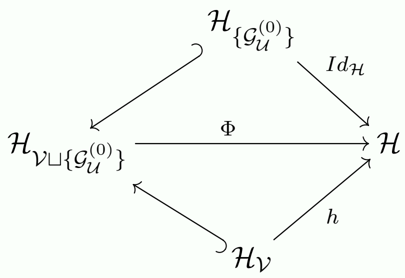
Computation of $[\psi_{\mathcal{U}}]\circ[\varphi]$:

Let's compute the pull-back 1-cocycle : $\mathcal{H}_{\{\mathcal{G}_{\mathcal{U}}^{(0)}\}} \xrightarrow{\varphi} \mathcal{G} \quad \mathcal{G}_{\mathcal{U}} \xrightarrow{\psi_{\mathcal{U}}} \mathcal{H}$.

The pull-back generalised covering is defined as $\{\mathcal{G}_{\mathcal{U}}^{(0)}\}^*_{\varphi}(\mathcal{U})_{0,j} = (\varphi^0)^{-1}(\mathcal{U}_j)$. Let's call it $\mathcal{V} = \{\mathcal{V}_j\} = \{(\varphi^0)^{-1}(\mathcal{U}_j)\}$.
Once again let $h : \mathcal{H}_{\mathcal{V}} \longrightarrow \mathcal{H}$ be the pull-back 1-cocycle, it is defined as:
$h^0((x,\mathcal{U}_k),\mathcal{V}_i) = \psi^0_{\mathcal{U}}(\varphi^0((x,\mathcal{U}_k),\mathcal{G}_{\mathcal{U}}^{(0)}),\mathcal{U}_i) = (x,\mathcal{U}_i)$ and
$h(\mathcal{V}_i,(\mathcal{U}_k,g,\mathcal{U}_l),\mathcal{V}_j) = \psi_{\mathcal{U}}(\mathcal{U}_i,\varphi(\mathcal{G}_{\mathcal{U}}^{(0)},(\mathcal{U}_k,g,\mathcal{U}_l),\mathcal{G}_{\mathcal{U}}^{(0)}),\mathcal{U}_j) = \psi_{\mathcal{U}}(\mathcal{U}_i,g,\mathcal{U}_j) = (\mathcal{U}_i,g,\mathcal{U}_j)$.
Then now we have two 1-cocycles to compare:

$$Id_{\mathcal{H}} : \begin{array}{ccc} \mathcal{H}_{\mathcal{G}_{\mathcal{U}}^{(0)}} & \longrightarrow & \mathcal{H} \\ (\mathcal{G}_{\mathcal{U}}^{(0)},(\mathcal{U}_k,g,\mathcal{G}_l),\mathcal{G}_{\mathcal{U}}^{(0)}) & \mapsto & (\mathcal{U}_k,g,\mathcal{U}_l) \\ ((x,\mathcal{U}_k),\mathcal{G}_{\mathcal{U}}^{(0)}) & \mapsto & (x,\mathcal{U}_k) \end{array} \; ; \quad h : \begin{array}{ccc} \mathcal{H}_{\mathcal{V}} & \longrightarrow & \mathcal{H} \\ (\mathcal{V}_i,(\mathcal{U}_k,g,\mathcal{U}_l),\mathcal{V}_j) & \mapsto & (\mathcal{U}_i,g,\mathcal{U}_j) \\ ((x,\mathcal{U}_k),\mathcal{V}_i) & \mapsto & (x,\mathcal{U}_i) \end{array}$$

Actually they are equivalent as we hope. To see it we must build a 1-cocycle Φ associated to the generalized covering $\mathcal{V}\sqcup\{\mathcal{G}_{\mathcal{U}}^{(0)}\}$ such that the following diagram



commutes.
There is only on way to define Ψ^0, and one way to define Ψ for the arrows whose ends are both labelled in $\mathcal{V}$ or both in $\{\mathcal{G}_{\mathcal{U}}^{(0)}\}$. Let's define Ψ for arrows which start with a label in $\{\mathcal{G}_{\mathcal{U}}^{(0)}\}$ and ends with the label in $\mathcal{V}$.
We set $\Phi(\mathcal{V}_j,(\mathcal{U}_k,g,\mathcal{U}_l),\mathcal{G}_{\mathcal{U}}^{(0)}) = (\mathcal{U}_j,g,\mathcal{U}_l)$.
Now, as usual, we will only check the functoriality according the source map only in this case:

$$\begin{aligned} s(\Phi(\mathcal{V}_j,(\mathcal{U}_k,g,\mathcal{U}_l),\mathcal{G}_{\mathcal{U}}^{(0)})) &= s(\mathcal{U}_j,g,\mathcal{U}_l) \\ &= (s(g),\mathcal{U}_l) \\ &= Id^0_{\mathcal{H}}((s(g),\mathcal{U}_l),\mathcal{G}_{\mathcal{U}}^{(0)}) \\ &= \Phi^0((s(g),\mathcal{U}_l),\mathcal{G}_{\mathcal{U}}^{(0)}) \\ &= \Phi^0(s(\mathcal{U}_k,g,\mathcal{U}_l),\mathcal{G}_{\mathcal{U}}^{(0)}) \\ &= \Phi^0(s(\mathcal{V}_j,(\mathcal{U}_k,g,\mathcal{U}_l),\mathcal{G}_{\mathcal{U}}^{(0)})). \end{aligned}$$

Then eventually $[Id_{\mathcal{H}}] = [h] = [\psi_{\mathcal{U}}] \circ [\varphi]$.

Then we can conclude that $\mathcal{G} \overset{\mathcal{M}}{\sim} \mathcal{G}_{\mathcal{U}}$.

Example 1.3.20 Topological manifolds and point

Let M be a smooth manifold, let $M \times M \rightrightarrows M$ it product groupoïd. Let also $\{\mathbb{1}_*\} \rightrightarrows \{*\}$ the trivial groupoïd with one point. We will show that these groupoïds are Morita equivalent : $M \times M \overset{\mathcal{M}}{\sim} \{\mathbb{1}_*\}$.
We consider the trivial generalised coverings $\{M\}$ and $\{\{*\}\}$ of M and $\{*\}$ and the Hilsum-Skandalis morphisms $[\varphi_{\{M\}}] : M \times M \dashrightarrow \{\mathbb{1}_*\}$ and $[\varphi_{\{\{*\}\}}] : \{\mathbb{1}_*\} \dashrightarrow M \times M$, induced by:

$$\varphi_{\{M\}} : \begin{array}{ccc} (M \times M)_{\{M\}} & \longrightarrow & \{\mathbb{1}_*\} \\ (M,(x,y),M) & \mapsto & \mathbb{1}_* \\ (x,M) & \mapsto & * \end{array} \quad \text{and} \quad \varphi_{\{\{*\}\}} : \begin{array}{ccc} \{\mathbb{1}_*\}_{\{\{*\}\}} & \longrightarrow & (M \times M) \\ (\{*\},\mathbb{1}_*,\{*\}) & \mapsto & (x_0,x_0) \\ (*,\{*\}) & \mapsto & x_0 \end{array}$$

, with $x_0 \in M$ a fixed point. As we did in the previous example, we now have to compute one representative of the composed equivalence class $[\varphi_{\{M\}}] \circ [\varphi_{\{\{*\}\}}]$ and $[\varphi_{\{\{*\}\}}] \circ \varphi_{\{M\}}]$, once again we compute the pull-back 1-cocycles.

<u>Computation of $[\varphi_{\{M\}}] \circ [\varphi_{\{\{*\}\}}]$</u>:
We can easily show that all the 1-cocycles with value in the groupoïd $\{\mathbb{1}_*\} \rightrightarrows \{*\}$ are equivalent (all objects and arrows will be sent to $*$ and $\mathbb{1}_*$ by every 1-cocycle, then they are compatible). In particular we have $[\varphi_{\{M\}}] \circ [\varphi_{\{\{*\}\}}] = [Id_{\mathbb{1}_*}]$.
<u>Computation of $[\varphi_{\{\{*\}\}}] \circ [\varphi_{\{M\}}]$</u> : We begin with the pull-back generalised covering <u>$\{M\}^*(\{\{*\}\}) = (\varphi^0_{\{M\}})^{-1}(\{*\}) \cap M$</u> $= M_{\{M\}} = \{(x,M) : x \in M\}$.
Then we can compute the pull-back 1-cocycle that we will call f:
$f^0(x,M) = \varphi^0_{\{\{*\}\}}(\varphi^0_{\{M\}}(x,M),\{*\}) = x_0$ and
$f(M,(x,y),M) = \varphi_{\{\{*\}\}}(\{*\},\varphi_{\{M\}}(M,(x,y),M),\{*\}) = (x_0,x_0)$.
Then now we have to show that these tow 1-cocycles are equivalent:

$$f : \begin{array}{ccc} (M \times M)_{\{M\}} & \longrightarrow & M \times M \\ (x,M) & \mapsto & x_0 \\ (M,(x,y),M) & \mapsto & (x_0,x_0) \end{array} \quad \text{and} \quad Id_{M\times M} : \begin{array}{ccc} (M \times M)_{\{\widetilde{M}\}} & \longrightarrow & M \times M \\ (x,\widetilde{M}) & \mapsto & x \\ (\widetilde{M},(x,y),\widetilde{M}) & \mapsto & (x,y) \end{array}$$

with $\widetilde{M} = M$, I write them differently to avoid confusions in the next disjoint union.
Once again to define a 1-cocycle $\Phi : (M \times M)_{M \sqcup \widetilde{M}} \longrightarrow M \times M$, such that the 1-cocycle equivalence diagram commutes, the only "not forced" part is the image of an arrow labeled with $\{M\}$ and $\{\widetilde{M}\}$ at the same time.
Let's define only in this case, we set:
$\Phi(M,(x,y),\widetilde{M}) = (x_0,y)$ and $\Phi(\widetilde{M},(x,y),M) = (x,x_0)$. Then the diagram will commute. Once again we can check the functoriality according the source map in one of these two cases.

$$\begin{aligned} s(\Phi(M,(x,y),\widetilde{M})) &= s(x_0,y) \\ &= y \\ &= Id_{M\times M}(y,\widetilde{M}) \\ &= \Phi^0(y,\widetilde{M}) \\ &= \Phi^0(s(M,(x,y),\widetilde{M})). \end{aligned}$$

Then $[\varphi_{\{\{*\}\}}] \circ [\varphi_{\{M\}}] = [f] = [Id_{M\times M}]$.
Then we can conclude that $M \times M \overset{\mathcal{M}}{\sim} \{\mathbb{1}_*\}$.

The notions we defined in this section will not be used in the rest of this document.

In the context of this document, we are interested in Lie groupoïds. As we said in the beginning of this part, Lie groupoïds have a natural equivalence relation induced by the usual notion of isomorphims between differential manifolds. Then our purpose in this part was to construct a notion of topological groupoïds equivalence from the usual notion of compatibles covering. Namely the Morita equivalence between topological groupoïds.
For this report, we will stop these investigations here, but to understand why the new category $Grpd_{HS}$ that we built, is more adapted to K-theoretical studies than the classical *Grpd*, let's spoil the next part of the Hilsum-Skandalis show.
It is possible to construct a notion of Morita equivalence between C^*-algebras such that if two locally compact topological groupoïds endowed with Haar systems are Morita equivalent, then their full C^*-algebras will be equivalent for this new Morita notion. The next point is that, with this new notion, two C^*-algrbras which are Morita equivalent have the same K-theory. It means that, if two groupoïds are Morita equivalent the way we defined above, then they have the same K-theoretic properties.

Then we can try to understand our two last examples with these considerations.
The first shows us that starting from a topological space endowed with charts (namely a manifold), we don't change K-theoretic properties by adding more and more charts. In such a way these extra charts are useless because they already "decompose" in the previous ones.
The second example is really interesting for this report because later we will have to understand what is the K-theory of the pair groupoïd $M \times M$. Using the previous considerations we can state that $K_0(C^*(M \times M)) \cong K_0(C^*(\{\mathbb{1}_*\})) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$. But we will also prove it with a different method using the C^*-algebras of compact operators on a Hilbert space. The relation $M \times M \overset{\mathcal{M}}{\sim} \{\mathbb{1}_*\}$ means that these groupoïds look alike in such a way, that their covering are compatibles. Let's take a look at their categorical structure (considering that we take their trivial covering). Then in the groupoïd $\{\mathbb{1}_*\}$, for each pair of objects (there is only $(*, *)$), there is a unique arrow from the first one to the second one. The product groupoïd $M \times M$ has the same property : to each pair of objects (x, y) there is a unique arrow from the first one to the second, which is the one we usually denote (y, x). This property make their coverings compatible.

Chapter 2

Deformation to the normal cone

2.1 Normal bundle

Definition 2.1.1 (Category of pairs)

We define the category $\mathcal{C}_2^\infty$ by fixing its objects to be the pairs of smooth (possibly non Hausdorff) manifolds (X, Y) where $Y \subset X$ is a closed embedded submanifold (closed in the topological sense, it could have boundary or being non compact).
And the morphisms between two pairs (X, Y) and (M, N) are smooth maps $f : X \longrightarrow M$ that map Y into N, we denote such a map by $f : (X, Y) \longrightarrow (M, N)$.

In this context, if Y is not set-theoretically included in X, but only embedded : $Y \overset{i}{\hookrightarrow} X$; then we will allow us to denote (X, Y) instead of $(X, i(Y))$ when the embedding is canonical. For instance, we will denote $(\mathbb{R}^n, \mathbb{R}^p)$ and not $(\mathbb{R}^n, \mathbb{R}^p \times \{0\}^{n-p})$.
The same for the elements of these space, in our constructions we will denote $y \in Y$ even if the element should be in $i(Y) \subset X$. For instance if we want to use the source map of a groupoïd $\mathcal{G}$, beacuse $\mathcal{G}^{(0)} \overset{u}{\hookrightarrow} \mathcal{G}$, then for $x \in \mathcal{G}^{(0)}$, by $s(x)$ we actualy mean $s(u(x)) = s(\mathbb{1}_x)$ which is x in this case.

Definition 2.1.2 (Adapted chart)

Let $(X, Y) \in \mathcal{C}_2^\infty$ of dimension (n, p) and (U, φ) a chart of X. This chart is said to be adapted to Y if $\varphi(U \cap Y) = \varphi(U) \cap (\mathbb{R}^p \times \{0\}^{n-p})$. Namely:

$$\forall\, (x_1, \dots, x_n) \in U; [(x_1, \dots, x_n) \in Y \text{ if and only if } \varphi(x_1, \dots, x_n) = (*_1, \dots, *_p, 0, \dots, 0)].$$

Definition 2.1.3 (Normal bundle)

Let $(X, Y) \in \mathcal{C}_2^\infty$. If we denote $TX|_Y$ the sub bundle $\sqcup_{y \in Y} T_y X$ then we can define the normal bundle of X when respect to Y by equivalent ways:

1. As the cokernel of the inclusion : $\mathcal{N}(X, Y) := Coker(TY \hookrightarrow TX|_Y) = TX|_Y / TY = \sqcup_{y \in Y} T_y X / T_y Y$
2. As a quotient bundle: $0 \longrightarrow TY \longrightarrow TX|_Y \longrightarrow \mathcal{N}(X, Y) \longrightarrow 0$

The idea of the normal bundle is the following: At each point of the submanifold Y, we look at the vectors tangent to X, and we remove the components which are tangent to Y. Then we get vectors tangent to X and normal to Y in a certain sense. This construction will allow us to classify the vectors of $TX|_Y$ to understand the way Y is embedded in X.

For now on, the normal bundle is only a topological space. We will endowed it with a manifold structure later. There is two usefull cases where its topology is easier to apprehend:

Now we will see functorial properties of the normal bundle.

Property 2.1.4

To a morphism $f : (X,Y) \longrightarrow (M,N)$ of the category $\mathcal{C}_2^\infty$ we can associate a map $d_\mathcal{N} f : \mathcal{N}(X,Y) \longrightarrow \mathcal{N}(M,N)$.

Proof

Let $f : (X,Y) \longrightarrow (M,N)$ morphism of the category of pairs of smooth manifolds. By differentiation and restriction to Y we get $\begin{array}{cccc} df|_Y : & TX|_Y & \longrightarrow & TM|_N \\ & (y,v) & \mapsto & (f(y), d_y f(v)) \end{array}$.

A quotient bundle is defined fiberwise, then it is enough to check that the maps $\begin{array}{cccc} d_y f : & T_y X & \longrightarrow & T_{f(y)}M \\ & v & \mapsto & d_y f(v) \end{array}$, $y \in Y$, passes to the quotient after composition with π the canonical projection on $T_{f(y)}M/T_{f(y)}N$.
By using the fact that f maps Y to N, we get that $T_yY \subset Ker(\pi \circ d_y f)$, then we are allowed to define the map from the quotient space. Then we put all fibers together to get the map:

$$\begin{array}{cccc} d_\mathcal{N} f : & \mathcal{N}(X,Y) & \longrightarrow & \mathcal{N}(M,N) \\ & (y, v \mod T_yY) & \mapsto & (f(y), d_y f(v) \mod T_{f(y)}N) \end{array}$$

Property 2.1.5

For every (X,Y) of $\mathcal{C}_2^\infty$, $\mathcal{N}(X,Y)$ is a manifold of dimension $dim\, X$.

Proof

As $\mathcal{N}(\mathbb{R}^n, \mathbb{R}^p) = T\mathbb{R}^n|_{\mathbb{R}^p \times \{0\}} / T(\mathbb{R}^p \times \{0\})|_{\mathbb{R}^p \times \{0\}} = \Big(\mathbb{R}^p \times \{0\}\Big) \times \Big(\mathbb{R}^n/(\mathbb{R}^p \times \{0\})\Big)$ which we can identify with $\mathbb{R}^p \times \mathbb{R}^{n-p} \cong \mathbb{R}^n$. Then from an adapted chart (U, φ) of (X,Y), using this identification, we can build a map: $(V := U \cap Y)$

$$\begin{array}{cccc} d_\mathcal{N}\varphi : & \mathcal{N}(U,V) & \longrightarrow & \mathbb{R}^n \\ & (x, v \mod T_xV) & \mapsto & (\underbrace{\varphi_1(x), ..., \varphi_p(x)}_{\in \mathbb{R}^p}, \underbrace{d_x\varphi_{p+1}(v), ..., d_x\varphi_n(v)}_{\in \mathbb{R}^{n-p}}) \end{array}$$

, with $\varphi = (\varphi_1, ..., \varphi_n)$.
To make it clear, when we use this identification we will write $\varphi = (\varphi^1, \varphi^2)$, and then $d_\mathcal{N}\varphi(x, v \mod T_xV) = (\varphi^1(x), d_x\varphi^2(v))$.
Because φ and $d\varphi$ are diffeomorphisms, this map is a diffeomorphism on its image (after identification).

Property 2.1.6

The map

$$\begin{array}{cccc} \mathcal{N}: & \mathcal{C}_2^\infty & \longrightarrow & \mathcal{C}^\infty \\ & (X,Y) & \mapsto & \mathcal{N}(X,Y) \\ & f & \mapsto & d_\mathcal{N} f \end{array}$$

define a covariant functor.
In particular we have $d_\mathcal{N} Id_{(X,Y)} = Id_{\mathcal{N}(X,Y)}$ and $d_\mathcal{N}(f \circ g) = d_\mathcal{N} f \circ d_\mathcal{N} g$.

Proof

We already saw that $\mathcal{N}(X,Y)$ is a smooth manifold, and $d_\mathcal{N} f$ is a diffeomorphism.
We can compute $d_\mathcal{N} Id_{(X,Y)}$ easily.
Let $f : (M,N) \to (A,B)$ and $g : (X,Y) \to (M,N)$ smooth pair morphisms. Then for $(y, v \mod T_yY)$ we have:

$$\begin{aligned} (d_\mathcal{N} f) \circ (d_\mathcal{N} g)(y, v \bmod T_yY) &= (d_\mathcal{N} f)(g(y), d_y g(v) \bmod T_{g(y)}N) \\ &= (f \circ g(y), d_{g(y)} f(d_y g(v)) \bmod T_{f \circ g(y)}B) \\ &= (f \circ g(y), d_y(f \circ g)(v) \bmod T_{f \circ g(y)}B) \\ &= d_\mathcal{N}(f \circ g)(y, v \bmod T_yY). \end{aligned}$$

Then $(d_\mathcal{N} f) \circ (d_\mathcal{N} g) = d_\mathcal{N}(f \circ g)$.

Remark 2.1.7

Until now, an element of $\mathcal{N}(X,Y)$ was denoted $(y, v \mod T_yY)$.
Because we checked that the maps are well defined through the quotient, we will allow us to denote only (y,v) considering that v already denote the equivalence class, when there is no risk of confusion.

Previously, to construct a chart of $\mathcal{N}(X,Y)$, we used a canonical identification : $\mathcal{N}(\mathbb{R}^n, \mathbb{R}^p) \cong \mathbb{R}^n$. In the next sections we will have to deal with two cases: the first one involves an analogue of the previous identification, and the second involves a non canonical one. We detail them now.

Let $* \in M$, we have:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T(M \times \{*\}) & \xrightarrow{i_1} & T(M \times M)|_{M \times \{*\}} & \xrightarrow{\pi} & \mathcal{N}(M \times M, M \times \{*\}) & \longrightarrow & 0 \\ & & \cong \downarrow j & & \| & & \downarrow \alpha_M & & \\ 0 & \longrightarrow & TM & \xrightarrow{i_2} & T(M \times M)|_{M \times \{*\}} & \xrightarrow{p} & TM & \longrightarrow & 0 \end{array}$$

with i_1 the canonical inclusion, π the canonical projection, $i_2(x,v) = ((x,*),(v,0))$, $p((x,*),(v,w)) = (x,w)$, $j((x,*),(v,0)) = (x,v)$ and $\alpha_M((x,*),(v,w) \mod T_{(x,*)}M \times \{*\}) = (x,w)$.

Rows are exact, squares commutes and j is an isomorphism, then using the fives lemma:

Property 2.1.8

If M is a manifold, then:

$$\begin{array}{llcll} \alpha_M: & \mathcal{N}(M\times M, M\times\{*\}) & \longrightarrow & \mathrm{TM} & \\ & ((x,*),(v,w) \mod T_{(x,*)}M\times\{*\}) & \mapsto & (x,w) & \end{array} \text{ is an isomorphism.}$$

If M is a manifold, we denote by Δ_M the diagonal submanifold $\{(x,x) : x\in M\}$ endowed with the obvious charts $(\Delta_U, \varphi\times\varphi)$, where (U,φ) is a chart of M.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T\Delta_M & \overset{i_1}{\longrightarrow} & T(M\times M)|_{\Delta_M} & \overset{\pi}{\longrightarrow} & \mathcal{N}(M\times M,\Delta_M) & \longrightarrow & 0 \\ & & \cong\downarrow j & & \| & & \downarrow \beta_M & & \\ 0 & \longrightarrow & TM & \overset{i_2}{\longrightarrow} & T(M\times M)|_{\Delta_M} & \overset{\delta}{\longrightarrow} & TM & \longrightarrow & 0 \end{array}$$

with i_1 the canonical inclusion, π the canonical projection, $i_2(x,v) = ((x,x),(v,v))$, $\delta((x,x),(v,w)) = (x, v-w)$, $j((x,x),(v,v)) = (x,v)$ and $\beta_M((x,x),(v,w) \mod T_{(x,x)}\Delta_M) = (x, v-w)$.
The rows are exact, the squares commutes and j is an isomorphism, then using the fives lemma:

Property 2.1.9

If M is a manifold, then:

$$\begin{array}{llcll} \beta_M: & \mathcal{N}(M\times M, \Delta_M) & \longrightarrow & \mathrm{TM} & \\ & ((x,x),(v,w) \mod T_{(x,x)}\Delta_M) & \mapsto & (x,v-w) & \end{array} \text{ is an isomorphism.}$$

2.2 Cone

In this section we use the normal bundle functor to define another functor called *deformation to the normal cone* or *DNC* and denoted $\mathscr{D}$.
Again this functor will start in the category $\mathcal{C}_2^\infty$ and ends in $\mathcal{C}^\infty$.

Here to define the "object side", we will starts by defining it as a set, then we will study the canonic case $(X,Y) = (\mathbb{R}^n, \mathbb{R}^p)$ to understand what the charts should be. And the last step will be to define the topology on our set using these charts to define a manifold.

Definition of the set: For $(X,Y)\in\mathcal{C}_2^\infty$, we set $D_Y^X = (\mathcal{N}(X,Y)\times\{0\}) \sqcup (X\times\mathbb{R}^*)$ without any topology.

Case $(X,Y) = (\mathbb{R}^n,\mathbb{R}^p)$: In this case we will be able to construct an atlas with only one chart. We will use the same identification of $\mathcal{N}(\mathbb{R}^n,\mathbb{R}^p)$ with $\mathbb{R}^n$ as we already did when we built normal bundle's charts.
Then the set $D_{\mathbb{R}^p}^{\mathbb{R}^n}$ could be idendified with $(\mathbb{R}^n\times\{0\})\sqcup(\mathbb{R}^n\times\mathbb{R}^*)$. For more intelligibility, we will denote it D_p^n.

Now we will define a topology on this space, for this purpose we will define a bijection between D_p^n and a topological space. We will define a subset of D_p^n to be open if and only of its image by the bijection is open. We will call this topology on D_p^n the *topology induced by the bijection*.

Let $q = n-p$ the codimension of $\mathbb{R}^p$ in $\mathbb{R}^n$.

$$\Psi : \begin{array}{ccc} \mathbb{R}^{p+q} \times \mathbb{R} & \longrightarrow & D_p^n \\ (y, \xi, t) & \mapsto & \begin{cases} (y, \xi, 0) & \text{if } t = 0 \\ (y, t\xi, t) & \text{if } t \neq 0 \end{cases} \end{array}$$

is bijective. Its reciprocal is obvious.

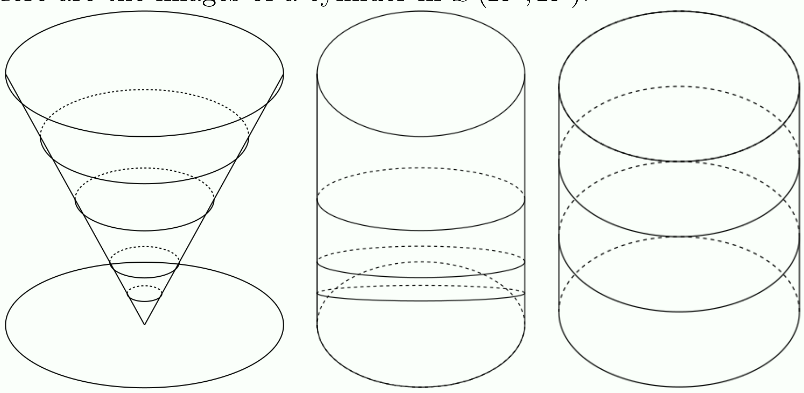
We denote $\mathscr{D}(\mathbb{R}^n, \mathbb{R}^p)$ the set D_p^n endowed with the topology induced by Ψ. By definition of the topology, Ψ and Ψ^{-1} put open sets on open sets. Then Ψ is an homeomorphism. Moreover on each point of both spaces, we can differentiate Ψ and Ψ^{-1} on a neighbourhood. Then Ψ is a diffeomorphism. Then

Property 2.2.1

For $0 \leqslant p \leqslant n$, $\mathscr{D}(\mathbb{R}^n, \mathbb{R}^p)$ is a $n+1$ dimensional manifold with one global chart, namely (D_p^n, Ψ).

Example 2.2.2 $n = 2$, $0 \leqslant p \leqslant 2$

Here are the images of a cylinder in $\mathscr{D}(\mathbb{R}^n, \mathbb{R}^p)$:



$p = 0$ $p = 1$ $p = 2$

The space $\mathscr{D}(\mathbb{R}^n, \mathbb{R}^p)$ with its differential structure will bu used to parametrize the general $\mathscr{D}(X, Y)$. In this purpose we should identify some open sets of $\mathscr{D}(\mathbb{R}^n, \mathbb{R}^p)$:

Let $U \subset \mathbb{R}^n$ open set, $V = U \cap \mathbb{R}^p$ the induced open set of $\mathbb{R}^p$. As we already did in the previous section with the global normal bundle $\mathcal{N}(\mathbb{R}^n, \mathbb{R}^p)$, we use the identification :

$$TU|_{V\times\{0\}}/T(V\times\{0\})|_{V\times\{0\}} = \big(V\times\{0\}\times\mathbb{R}^n\big)/\big(V\times\{0\}\times\mathbb{R}^p\times\{0\}\big) = V\times\{0\}\times(\mathbb{R}^n/(\mathbb{R}^p\times\{0\})) \cong V\times\mathbb{R}^q.$$

We recall that a bundle quotient consists of a fiberwise quotient.

Then we set $\mathcal{N}(U, V) = V \times \mathbb{R}^q$, and set theoretically we get $D_V^U = (V \times \mathbb{R}^q \times \{0\}) \sqcup (U \times \mathbb{R}^*) \subset D_p^n$. But $D_V^U = \Psi(\Omega_V^U)$ where

$$\Omega_V^U = f^{-1}(U) \text{ with } f : \begin{array}{ccc} \mathbb{R}^{p+q} \times \mathbb{R} & \longrightarrow & \mathbb{R}^{p+q} \\ (y, \xi, t) & \mapsto & (y, t\xi) \end{array} \text{ continuous map.}$$

Then D_V^U is open in $\mathscr{D}(\mathbb{R}^n, \mathbb{R}^p)$. We denote $\mathscr{D}(U, V)$ the set D_V^U endowed with the topology induced by $\mathscr{D}(\mathbb{R}^n, \mathbb{R}^p)$.

Now we use $\mathscr{D}(\mathbb{R}^n, \mathbb{R}^p)$ and the open sets $\mathscr{D}(U, V)$ to parametrize a general $\mathscr{D}(X, Y)$.

General case: Let $(X, Y) \in \mathcal{C}_2^\infty$, let (U, φ) a chart of X adapted to Y. $V = U \cap Y$ the induced open set of Y. Set theoretically we have the inclusion $D_V^U \subset D_Y^X$. The set D_V^U will play the role of an open set of D_Y^X. We set:

$$\begin{array}{cccc} \mathscr{D}(\varphi) & D_V^U & \longrightarrow & \mathscr{D}(\varphi(U), \varphi(V)) \subset \mathscr{D}(\mathbb{R}^n, \mathbb{R}^p) \\ & z & \mapsto & \begin{cases} (d_{\mathcal{N}}\varphi(y, v), 0) & \text{if } z = (y, v, 0) \\ (\varphi(x), t) & \text{if } z = (x, t) \end{cases}, \quad \text{which is a bijective map.} \end{array}$$

Indeed we get the reciprocal by stating a map $\mathscr{D}(\varphi^{-1})$ the same way.
Then we define a topology on D_V^U by defining a subset to be open if and only if its image by $\mathscr{D}(\varphi)$ is open in $\mathscr{D}(\varphi(V), \varphi(U)) \subset \mathscr{D}(\mathbb{R}^n, \mathbb{R}^p)$.
Then in particular, D_V^U is open, and we denote it $\mathscr{D}(U, V)$ when endowed with this topology.

The map $\mathscr{D}(\varphi)$ then is a diffeomorphism, and then $(D_V^U, \mathscr{D}(\varphi))$ is a chart of D_Y^X

If $\{(U_i, \varphi_i)\}_i$ is an atlas of X with respect to Y, we denote by $\mathscr{D}(X, Y)$ the space D_Y^X endowed with the differential manifold structure induced by $\{(D_{U_i \cap Y}^{U_i}, \mathscr{D}(\varphi_i))\}_i$.

We can summerize it saying that:

Property 2.2.3

If $(X, Y) \in \mathcal{C}_2^\infty$ is a pair of manifold, then $\mathscr{D}(X, Y)$ is a smooth manifold. There is a maximal atlas which contains the charts of the form $(\mathscr{D}(U, V), \psi^{-1} \circ \mathscr{D}(\varphi))$ where $V = U \cap Y$ and (U, φ) is a chart of X adapted to Y.

This property gives the "object side" of the functor $\mathscr{D}$.

Actually the functoriality of $\mathscr{D}$ is a consequence of the functoriality of $\mathcal{N}$. Using what we know on the normal bundle, we can generalize what we did to build the charts of $\mathscr{D}(X, Y)$.

Property 2.2.4

For $f : (X, Y) \to (M, N)$ morphism of the category $\mathcal{C}_2^\infty$, we define:

$$\begin{array}{ccccl} \mathscr{D}(f): & \mathscr{D}(X, Y) & \longrightarrow & \mathscr{D}(M, N) & \\ & (y, v, 0) & \mapsto & d_{\mathcal{N}} f(y, v) & |(y, v) \in \mathcal{N}(X, Y) \\ & (x, t) & \mapsto & (f(x), t) & |t \neq 0,\, x \in X \end{array}$$

Using the same argument as before, this map $\mathscr{D}(f)$ is a diffeomorphism. Using the functoriality of the normal bundle functor, we can easily deduce that in general $\mathscr{D}(f \circ g) = \mathscr{D}(f) \circ \mathscr{D}(g)$ and $\mathscr{D}(Id_{(X,Y)}) = Id_{\mathscr{D}(X,Y)}$. Then the "morphism side" of $\mathscr{D}$ is well defined.

Then:

Property 2.2.5 Normal cone functoriality

$$\begin{array}{cccc} \mathscr{D}: & \mathcal{C}_2^\infty & \longrightarrow & \mathcal{C}^\infty \\ & (X, Y) & \mapsto & \mathscr{D}(X, Y) \quad \text{is a functor.} \\ & f & \mapsto & \mathscr{D}(f) \end{array}$$

We will end this part by defining an identification which will be usefull later. It corresponds to the case $M = N$ with the previous notations.

Property 2.2.6

If $X \in \mathcal{C}_2^\infty$, then the quotient bundle $\mathcal{N}(X,X)$ becomes: $\mathcal{N}(X,X) = TX_{|X}/TX = TX/TX = X \times \{0\} \overset{\overline{n}}{\cong} X$. Then calling $\overline{n}$ this isomorphism, we can identify $\mathcal{N}(X,X)$ with X.

Because X has a codimension $q = 0$, there is no cone normal deformation and using the identification we get: $\mathscr{D}(X,X) = (\mathcal{N}(X,X)\times\{0\}) \sqcup (X \times \mathbb{R}^*) \cong (X\times\{0\})\sqcup(X\times\mathbb{R}^*) = X\times\mathbb{R}$. We will also denote $\overline{n}$ this last isomorphism.

Now if we have a morphism of pairs $f : (M,N) \to (X,X)$ then $d_{\mathcal{N}}f : (x, v \mod T_xN) \mapsto (f(x), d_xf(v) \mod T_{f(x)}X) = (f(x), 0 \mod T_{f(x)}X)$ can be identified with the map :
$$\begin{array}{ccc} \mathcal{N}(M,N) & \longrightarrow & X \\ (x, v \mod T_xN) & \mapsto & f(x) \end{array}.$$ We will denote it $\overline{n}\circ d_{\mathcal{N}}f$.

Using the same identification we can define:
$$\begin{array}{cccl} \mathscr{D}(M,N) & \to & X\times\mathbb{R} & \\ (y,v,0) & \mapsto & (f(y),0) & |(y,v)\in\mathcal{N}(M,N) \\ (x,t) & \mapsto & (f(x),t) & |t\neq 0,\, x\in M \end{array}.$$ We will denote it $\overline{n}\circ\mathscr{D}(f)$

2.3 Example of explicit calculus

In this section we will compute an open set of $\mathscr{D}(\mathbb{S}^1, \{*\})$ where $*$ is a point of $\mathbb{S}^1$, say $(1,0)$. We take the open set $U = \mathbb{S}^1 \backslash \{*\}$ and $\begin{array}{cccc} \eta: &]-\pi,\pi[& \to & U\subset\mathbb{R}^2 \\ & \theta & \mapsto & (\cos\theta,\sin\theta) \end{array}$ one of its parametrization. Then, if we set $\varphi = \eta^{-1}$, we get a chart (U,φ).

Let's compute the normal bundle : $\mathcal{N}(U,\{*\}) = T_*U/T_*\{*\} = T_*U = \{*\}\times\mathbb{R}$.

Now we compute the normal differential $d_{\mathcal{N}}\varphi$: We have $\begin{array}{cccc} d_0\eta: &]-\pi,\pi[& \to & T_*U\subset T_*\mathbb{S}^1 \\ & \lambda & \mapsto & \begin{pmatrix}0\\1\end{pmatrix}.\ (\lambda) = \begin{pmatrix}0\\ \lambda\end{pmatrix} \end{array}$.

Its inverse gives us

$$\boxed{\begin{array}{cccc} d_*\varphi: & T_*U & \to & T_0]-\pi,\pi[\\ & (0,\lambda) & \mapsto & \lambda \end{array}}$$ which provide us, using the identification :

$$\boxed{\begin{array}{cccc} d_{\mathcal{N}}\varphi: & \mathcal{N}(U,\{*\}) & \to & \mathcal{N}(\mathbb{R},\{0\}) \\ & ((1,0),(0,\lambda)) & \mapsto & \lambda \end{array}}$$

Then we can define the topology of the cone $D^U_{\{*\}} : (\{*\}\times\mathbb{R}\times\{0\}) \sqcup (U\times\mathbb{R}^*)$
We have:

$$\begin{array}{cccc} \mathscr{D}(\varphi): & D^U_{\{*\}} & \longrightarrow & D^{]-\pi,\pi[}_{\{0\}} \subset D^1_0 \\ & (*,(0,\lambda),0) & \mapsto & (\lambda,0) \\ & (\cos\theta,\sin\theta,t) & \mapsto & (\theta,t) \end{array}$$

Then $(\Psi^{-1}\circ\mathscr{D}(\varphi))^{-1} = \mathscr{D}(\eta)\circ\Psi$ gives us a parametrization of $\mathscr{D}^U_{\{*\}}$:

$$\boxed{\begin{array}{cccccc} \mathscr{D}(\eta)\circ\Psi: & \mathbb{R}\times\mathbb{R} & \longrightarrow & \mathscr{D}^1_0 & \longrightarrow & \mathscr{D}^U_{\{*\}} \\ & (\xi,0) & \mapsto & (\xi,0) & \mapsto & (*,(0,\xi),0) \\ & (\xi,t) & \mapsto & (t\xi,t) & \mapsto & (\cos(t\xi),\sin(t\xi),t) \end{array}}$$

Chapter 3

Connes's tangent groupoïd

3.1 General definitions

Now our goal is the following, starting from a Lie groupoïd $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$, we want to construct a Lie groupoïd of the form $\mathscr{D}(\mathcal{G}, \mathcal{G}^{(0)}) \rightrightarrows \mathcal{G}^{(0)} \times \mathbb{R}$.

Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ a Lie groupoïd, and s, t, m, u, i its structural maps.

<u>Source and target</u> :

Recalling the canonical embedding $u : \mathcal{G}^{(0)} \hookrightarrow \mathcal{G}$, we can see that s and t are actually pair morphisms $s, t : (\mathcal{G}, \mathcal{G}^{(0)}) \to (\mathcal{G}^{(0)}, \mathcal{G}^{(0)})$. Then we can consider the maps $\overline{n} \circ \mathscr{D}(s)$:

$$\boxed{\begin{array}{cccl} \mathscr{D}(\mathcal{G}, \mathcal{G}^{(0)}) & \to & \mathcal{G}^{(0)} \times \mathbb{R} & \\ (x, v, 0) & \mapsto & (s(\mathbb{1}_x), 0) = (x, 0) & |(x, v) \in \mathcal{N}(\mathcal{G}, \mathcal{G}^{(0)}) \\ (g, t) & \mapsto & (s(g), t) & |t \neq 0,\ g \in \mathcal{G} \end{array}}\ .$$

and $\overline{n} \circ \mathscr{D}(t)$ is defined the same way.
We call these maps s^{ad} and t^{ad}.

<u>Product</u>:

On the same model we will consider $\mathscr{D}(m)$ and use a canonical isomorphism to define our composition m^{ad}:

We will consider the isomorphism

$$\begin{array}{cccc} \Phi : & \mathcal{N}(\mathcal{G}, \mathcal{G}^{(0)})^{(2)} & \xrightarrow{\cong} & \mathcal{N}(\mathcal{G}^{(2)}, \mathcal{G}^{(0)}) \\ & ((x, v \mod T_x\mathcal{G}^{(0)}), (x, w \mod T_x\mathcal{G}^{(0)})) & \mapsto & ((x, x), (v, w) \mod T_{(x,x)}\Delta\mathcal{G}^{(0)}) \end{array}.$$

For the groupoïd $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$, the composition of arrows $m : (\mathcal{G}^{(2)}, \mathcal{G}^{(0)}) \longrightarrow (\mathcal{G}, \mathcal{G}^{(0)})$ is a map of pairs. (We consider $\mathcal{G}^{(0)}$ to be canonically injected in $\mathcal{G}^{(2)}$ by the map $\Delta \circ u : x \mapsto (\mathbb{1}_x, \mathbb{1}_x)$). Then we get the normal differential:

$$\begin{array}{cccc} d_{\mathcal{N}}m : & \mathcal{N}(\mathcal{G}^{(2)}, \mathcal{G}^{(0)}) & \longrightarrow & \mathcal{N}(\mathcal{G}, \mathcal{G}^{(0)}) \\ & ((x, x), (v, w) \mod T_{(x,x)}\Delta\mathcal{G}^{(0)}) & \mapsto & (x, d_{(x,x)}m(v, w) \mod T_{(x,x)}\Delta\mathcal{G}^{(0)}) \end{array}$$

Then let's compute $d_{(x,x)}m(v, w)$:
Let

$$\begin{array}{cccc} p_s : & (\mathcal{G}, \mathcal{G}^{(0)}) & \longrightarrow & (\mathcal{G}^{(2)}, \mathcal{G}^{(0)}) \\ & g & \mapsto & (g, \mathbb{1}_{s(g)}) \end{array} \qquad \begin{array}{cccc} p_t : & (\mathcal{G}, \mathcal{G}^{(0)}) & \longrightarrow & (\mathcal{G}^{(2)}, \mathcal{G}^{(0)}) \\ & g & \mapsto & (\mathbb{1}_{t(g)}, g) \end{array}$$

We have $d_{\mathcal{N}}p_s(x, v \mod T_x\mathcal{G}^{(0)}) = (p_s(x), d_xp_s(v) \mod T_{p_s(x)}\Delta\mathcal{G}^{(0)})$, with $p_s(x) = (x,x)$ and $d_xp_s(v) = d_x(Id_{\mathcal{G}} \times (u \circ s))(v) = (d_xId_{\mathcal{G}}(v), d_x(u \circ s)(v)) = (v, d_x(u \circ s)(v))$.
But $d_x(u \circ s)(v) \in T_x\mathcal{G}^{(0)}$. Then $d_{\mathcal{N}}p_s(x, v \mod T_x\mathcal{G}^{(0)}) = ((x,x),(v \mod T_x\mathcal{G}^{(0)}, 0 \mod T_x\mathcal{G}^{(0)})$.
And using the same process we have $d_{\mathcal{N}}p_t(x, v \mod T_x\mathcal{G}^{(0)}) = ((x,x),(0 \mod T_x\mathcal{G}^{(0)}, v \mod T_x\mathcal{G}^{(0)})$.

Then $m \circ p_s = Id_{\mathcal{G}} = m \circ p_t \quad \Longrightarrow \quad d_{\mathcal{N}}m \circ d_{\mathcal{N}}p_s = Id_{\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})} = d_{\mathcal{N}}m \circ d_{\mathcal{N}}p_t$.
The left equality gives us:

$$\begin{aligned}(x, v \mod T_x\mathcal{G}^{(0)}) &= Id_{\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})}(x, v \mod T_x\mathcal{G}^{(0)}) \\ &= d_{\mathcal{N}}m((x,x),(v \mod T_x\mathcal{G}^{(0)}, 0 \mod T_x\mathcal{G}^{(0)})) \\ &= (x, d_{(x,x)}m(v \mod T_x\mathcal{G}^{(0)}, 0 \mod T_x\mathcal{G}^{(0)}))\end{aligned}$$

Then $d_{(x,x)}m(v \mod T_x\mathcal{G}^{(0)}, 0 \mod T_x\mathcal{G}^{(0)}) = v \mod T_x\mathcal{G}^{(0)}$.
The same way, the right equality gives us $d_{(x,x)}m(0 \mod T_x\mathcal{G}^{(0)}, w \mod T_x\mathcal{G}^{(0)}) = w \mod T_x\mathcal{G}^{(0)}$.

Then we are able to compute the normal differential:

$$\begin{array}{rccc} d_{\mathcal{N}}m : & \mathcal{N}(\mathcal{G}^{(2)},\mathcal{G}^{(0)}) & \longrightarrow & \mathcal{N}(\mathcal{G},\mathcal{G}^{(0)}) \\ & ((x,x),(v,w) \mod T_{(x,x)}\Delta\mathcal{G}^{(0)}) & \mapsto & (x, v+w \mod T_{(x,x)}\Delta\mathcal{G}^{(0)}) \end{array}.$$

And then we have:

$$\begin{array}{rccc} \mathscr{D}(m) : & \mathscr{D}(\mathcal{G}^{(2)},\mathcal{G}^{(0)}) & \longrightarrow & \mathscr{D}(\mathcal{G},\mathcal{G}^{(0)}) \\ & ((x,x),(v,w) \mod T_{(x,x)}\Delta\mathcal{G}^{(0)}, 0) & \mapsto & (x, v+w \mod T_{(x,x)}\Delta\mathcal{G}^{(0)}, 0) \\ & ((g,h),t) & \mapsto & (gh,t) \end{array}$$

But what we need is to define a product on elements of $\mathscr{D}(\mathcal{G},\mathcal{G}^{(0)})$, then we need a map from $\mathscr{D}(\mathcal{G},\mathcal{G}^{(0)})^{(2)}$. Let's extend the isomorphism Φ to the deformation to the normal cone:

$$\begin{aligned}\mathscr{D}(\mathcal{G},\mathcal{G}^{(0)})^{(2)} &= \mathscr{D}(\mathcal{G},\mathcal{G}^{(0)}) \ {}_{s^{ad}}\!\!\underset{\mathcal{G}^{(0)}\times\mathbb{R}}{\times}\!\!{}_{t^{ad}} \ \mathscr{D}(\mathcal{G},\mathcal{G}^{(0)}) \\ &= (\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)}) \times \{0\} \sqcup \mathcal{G} \times \mathbb{R}^*) \ {}_{s^{ad}}\!\!\underset{\mathcal{G}^{(0)}\times\mathbb{R}}{\times}\!\!{}_{t^{ad}} \ (\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)}) \times \{0\} \sqcup \mathcal{G} \times \mathbb{R}^*) \\ \textit{(slice preserved)} &= \underbrace{\left[(\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)}) \times \{0\}) \ {}_{s^{ad}}\!\!\underset{\mathcal{G}^{(0)}\times\mathbb{R}}{\times}\!\!{}_{t^{ad}} \ (\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)}) \times \{0\})\right]}_{\overset{\Phi}{\cong} \ \mathcal{N}(\mathcal{G}^{(2)},\mathcal{G}^{(0)})\times\{0\}} \\ &\quad \sqcup \underbrace{\left[(\mathcal{G} \times \mathbb{R}^*) \ {}_{s^{ad}}\!\!\underset{\mathcal{G}^{(0)}\times\mathbb{R}}{\times}\!\!{}_{t^{ad}} \ (\mathcal{G} \times \mathbb{R}^*)\right]}_{\overset{\Phi}{\cong} \ \mathcal{G}^{(2)}\times\mathbb{R}^*}.\end{aligned}$$

Then we get a new isomorphism $\overline{\Phi}$ defined as:

$$\begin{array}{rccc} \overline{\Phi} : & \mathscr{D}(\mathcal{G},\mathcal{G}^{(0)})^{(2)} & \longrightarrow & \mathscr{D}(\mathcal{G}^{(2)},\mathcal{G}^{(0)}) \\ & ((x, v \mod T_x\mathcal{G}^{(0)}, 0),(x, w \mod T_x\mathcal{G}^{(0)}, 0)) & \mapsto & ((x,x),(v,w) \mod T_{(x,x)}\Delta\mathcal{G}^{(0)}, 0) \\ & ((g,t),(h,t)) & \mapsto & ((g,h),t). \end{array}$$

Eventually we set $m^{ad} = \mathscr{D}(m) \circ \overline{\Phi}$, we get:

$$\boxed{\begin{array}{rccc} m^{ad} : & \mathscr{D}(\mathcal{G},\mathcal{G}^{(0)})^{(2)} & \longrightarrow & \mathscr{D}(\mathcal{G},\mathcal{G}^{(0)}) \\ & ((x,v,0),(x,w,0)) & \mapsto & (x, v+w, 0) \\ & ((g,t),(h,t)) & \mapsto & (gh,t) \end{array}}$$

Unitarization:

The same way we see that the maps $u : \mathcal{G}^{(0)} \to \mathcal{G}$ and $i : \mathcal{G} \to \mathcal{G}$ are moprphisms of pair of manifolds : $u : (\mathcal{G}^{(0)}, \mathcal{G}^{(0)}) \to (\mathcal{G}, \mathcal{G}^{(0)})$ and $i : (\mathcal{G}, \mathcal{G}^{(0)}) \to (\mathcal{G}, \mathcal{G}^{(0)})$. Then we get:

$$\begin{array}{rccc} d_{\mathcal{N}} u \circ \overline{n}^{-1} : & \mathcal{G}^{(0)} & \to & \mathcal{N}(\mathcal{G}, \mathcal{G}^{(0)}) \\ & x & \mapsto & (\mathbb{1}_x, d_x u(0) \mod T_{\mathbb{1}_x} \mathcal{G}^{(0)}) = (\mathbb{1}_x, 0) \end{array}$$

Passing to the normal cone we get what we will denote u^{ad}:

$$\boxed{\begin{array}{rccc} \mathscr{D}(u) \circ \overline{n}^{-1} = u^{ad} : & \mathcal{G}^{(0)} \times \mathbb{R} & \to & \mathscr{D}(\mathcal{G}, \mathcal{G}^{(0)}) \\ & (x, 0) & \mapsto & (\mathbb{1}_x, 0, 0) \\ & (x, t) & \mapsto & (\mathbb{1}_x, t), \ t \neq 0 \end{array}}$$

Inversion:

Eventually we set i^{ad} to be:

$$\boxed{\begin{array}{rccc} i^{ad} = \mathscr{D}(i) : & \mathscr{D}(\mathcal{G}, \mathcal{G}^{(0)}) & \to & \mathscr{D}(\mathcal{G}, \mathcal{G}^{(0)}) \\ & (x, v, 0) & \mapsto & (x, -v, 0) \\ & (g, t) & \mapsto & (g^{-1}, t) \end{array}}$$

We used these maps to define a new groupoïd:

Definition 3.1.1 Adiabatic groupoïd

Let $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ a Lie groupoïd with its structural maps s, t, m, u, i.

Then we call adiabatic groupoïd of $\mathcal{G}$ the groupoïd $\mathscr{D}(\mathcal{G}, \mathcal{G}^{(0)}) \rightrightarrows \mathcal{G}^{(0)} \times \mathbb{R}$ endowned with the structural maps $s^{ad}, t^{ad}, m^{ad}, u^{ad}, i^{ad}$ defined above.

Property 3.1.2

The adiabatic groupoïd associatied to a Lie groupoïd is also a Lie groupoïd.

Proof

$\mathscr{D}(\mathcal{G}, \mathcal{G}^{(0)})$ and $\mathcal{G}^{(0)} \times \mathbb{R}$ are smooth manifolds.
$\mathcal{G}^{(0)} \times \mathbb{R}$ is Hausdorff beause $\mathcal{G}^{(0)}$ is Hausdorff.
The maps $s^{ad}, t^{ad}, m^{ad}, u^{ad}$ are smooth beacuse of the functoriality of $\mathscr{D}$ and beacuse $\overline{n}$ and $\overline{\Phi}$ are smooth diffeomorphisms. Also using the functor $\mathscr{D}$, i being a diffeomorphism, then $i^{ad} = \mathscr{D}(i)$ is a diffeomorphism too.
The map s^{ad} is a submersion. (Admitted here).

We can see that in the adiabatic groupoïd, every object or arrow has a component $t \in \mathbb{R}$ which show us on which "slice" of the deformation we are. Our purpose being to study the deforation near the 0-slice, and the component $t \in \mathbb{R}$ being preserved by the structural maps, we can restrict the adiabatic groupoïd to the slices indexed by $t \in [0, 1]$.

Definition 3.1.3 Tangent groupoïd

If $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ is a Lie groupoïd, then we call *Tangent groupoïd* the Lie groupoïd got by restriction of the adiabatic groupoïd to the slices indexed by [0, 1]. We denote it $\mathcal{G}^T$

It has the form of $\left(\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)}) \times \{0\} \sqcup \mathcal{G}\times]0,1]\right) \rightrightarrows \left(\mathcal{G}^{(0)} \times [0,1]\right)$ endowed with the topology induced by the deformation to the normal cone one.

In such a way, the tangent groupoïd encode the deformation that we defined geometrically in a deformation of Lie groupoïds. This deformation takes place from the "$t = 1$" slice, which corresponds to the groupoïd $\mathcal{G} \rightrightarrows \mathcal{G}^{(0)}$ until the "$t = 0$" slice which corresponds to the normal bundle groupoïd $\mathcal{N}(\mathcal{G}^{(0)},\mathcal{G}) \rightrightarrows \mathcal{G}^{(0)}$.
Now our next step will be to build C^*-algebras from these groupoïds in order to use K-theory on these algebras to reveal links between them.

We will show it in details for reduced algebras which are more delicate to manipulate than the full ones. Then we will state the same properties for the full algebras without a detailed proof, the arguments being the same.

We begin algebraically with algebras of compactly supported continuous functions. We set:

$$ev_0 : \begin{array}{ccc} \mathcal{C}_c\left(\mathcal{G}^T\right) & \longrightarrow & \mathcal{C}_c\left(\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})\right) \\ f & \mapsto & f_{|\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})} \end{array} \quad \text{and} \quad ev_1 : \begin{array}{ccc} \mathcal{C}_c\left(\mathcal{G}^T\right) & \longrightarrow & \mathcal{C}_c\left(\mathcal{G}\right) \\ f & \mapsto & f_{|\mathcal{G}} \end{array}.$$

The reduced norms of the algebras $\mathcal{C}_c\left(\mathcal{G}^T\right)$, $\mathcal{C}_c\left(\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})\right)$ and $\mathcal{C}_c(\mathcal{G})$ will be denoted respectively $\|.\|_{r,\mathcal{G}^T}$, $\|.\|_{r,\mathcal{N}}$ and $\|.\|_{r,\mathcal{G}}$

Now we pre compose these maps with the inclusion in their completion by the reduced norm. Then we get the following maps whose we show their continuity.

Property 3.1.4

The maps:

$$\begin{array}{ccc} \left(\mathcal{C}_c\left(\mathcal{G}^T\right), \|.\|_{r,\mathcal{G}^T}\right) & \longrightarrow & \left(C_r^*(\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)}), \|.\|_{r,\mathcal{N}}\right) \\ f & \mapsto & f_{|\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})} \\ \left(\mathcal{C}_c\left(\mathcal{G}^T\right), \|.\|_{r,\mathcal{G}^T}\right) & \longrightarrow & \left(C_r^*(\mathcal{G}), \|.\|_{r,\mathcal{G}}\right) \\ f & \mapsto & f_{|\mathcal{G}} \end{array}$$

are continuous.

Proof

To construct the different reduced norms, we need families of representation: one family for each norm. We denote them $\left(\pi^{\mathcal{G}^T}_{(x,t)}\right)_{(x,t)\in\mathcal{G}^{(0)}\times[0,1]}$, $\left(\pi^{\mathcal{N}}_x\right)_{x\in\mathcal{G}^{(0)}}$ and $\left(\pi^{\mathcal{G}}_x\right)_{x\in\mathcal{G}^{(0)}}$.
Let $f \in \mathcal{C}_c\left(\mathcal{G}^T\right)$. Then the operator $\pi^{\mathcal{G}^T}_{(x,t)}(f)$ acts by convolution with f on the space $L^2\left((\mathcal{G}^T)^{(x,t)}\right)$ of square integrable functions defined on the target fiber associated to (t,x).
But $(\mathcal{G}^T)^{(x,t)} = \begin{cases} \mathcal{G}^x \times \{t\} & \text{if } t \neq 0 \\ \{x\} \times \mathcal{N}_x \times \{0\} & \text{if } t = 0,\ \mathcal{N}_x \text{ is the normal fiber associated to } x. \end{cases}$
Then comparing the maps:

$$\begin{array}{cccc} \pi^{\mathcal{G}^T}_{(x,1)}(f): & L^2(\mathcal{G}^x\times\{1\}) & \rightarrow & L^2(\mathcal{G}^x\times\{1\}) \\ & g & \mapsto & g*f \end{array} \; ; \; \begin{array}{cccc} \pi^{\mathcal{G}}_x(f_{|\mathcal{G}}): & L^2(\mathcal{G}^x) & \rightarrow & L^2(\mathcal{G}^x) \\ & g & \mapsto & g*f \end{array}$$

We see that their subordinate norms are equal : $\|\pi^{\mathcal{G}^T}_{(x,1)}(f)\| = \|\pi^{\mathcal{G}}_x(f_{|\mathcal{G}})\|$.
The same way with $t=0$ we get $\|\pi^{\mathcal{G}^T}_{(x,0)}(f)\| = \|\pi^{\mathcal{N}}_x(f_{|\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})})\|$.
Then we can conclude that:

$$\|f_{|\mathcal{G}}\|_{r,\mathcal{G}} = \sup_{x\in\mathcal{G}^{(0)}} \|\pi^{\mathcal{G}}_x(f_{|\mathcal{G}})\| = \sup_{x\in\mathcal{G}^{(0)}} \|\pi^{\mathcal{G}^T}_{(x,1)}(f)\| \leqslant \sup_{(x,t)\in\mathcal{G}^{(0)}\times[0,1]} \|\pi^{\mathcal{G}^T}_{(x,t)}(f)\| = \|f\|_{r,\mathcal{G}^T}$$

$$\|f_{|\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})}\|_{r,\mathcal{N}} = \sup_{x\in\mathcal{G}^{(0)}} \|\pi^{\mathcal{N}}_x(f_{|\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})})\| = \sup_{x\in\mathcal{G}^{(0)}} \|\pi^{\mathcal{G}^T}_{(x,0)}(f)\| \leqslant \sup_{(x,t)\in\mathcal{G}^{(0)}\times[0,1]} \|\pi^{\mathcal{G}^T}_{(x,t)}(f)\| = \|f\|_{r,\mathcal{G}^T}$$

Using this continuity, we are now able to extend these maps to the reduced C^*-algebra. We call these maps ev^r_0 and ev^r_1.

Property 3.1.5

The two previous maps extends as continous maps with the same subordinate norms:

$$\begin{array}{cccc} ev^r_0: & C^*_r(\mathcal{G}^T) & \longrightarrow & C^*_r(\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})) \\ & f & \mapsto & f_{|\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})} \end{array} \text{ and } \begin{array}{cccc} ev^r_1: & C^*_r(\mathcal{G}^T) & \longrightarrow & C^*_r(\mathcal{G}) \\ & f & \mapsto & f_{|\mathcal{G}} \end{array}.$$

Proof

We use the extension theorem presented in appendix:
The arrival spaces are complete (they were built ase a completion).
By definition $\mathcal{C}_c\left(\mathcal{G}^T\right)$ is dense in $C^*_r(\mathcal{G}^T)$ for the reduced norm.
The maps are continuous.

The set $\mathcal{G}\times]0,1]$ is open in $\mathcal{G}^T$, then there is a canonical injection $i_0 : \mathcal{C}_c\left(\mathcal{G}\times]0,1]\right) \hookrightarrow \mathcal{C}_c\left(\mathcal{G}^T\right)$ obtained by extending the function with zeros.
We now want to extend this map to the reduced C^*-algebras:

Property 3.1.6

The canonical injection i_0 extend to a continuous map $i^r_0 : C^*_r(\mathcal{G}\times]0,1]) \hookrightarrow C^*_r(\mathcal{G}^T)$.

Proof

Following the same extension process as before, we only need the map $\left(\mathcal{C}_c\left(\mathcal{G}\times]0,1]\right), \|.\|_{r,\mathcal{G}\times]0,1]}\right) \hookrightarrow (C^*_r(\mathcal{G}^T), \|.\|_{r,\mathcal{G}^T})$ to be continuous.
Let $f \in \mathcal{C}_c\left(\mathcal{G}\times]0,1]\right)$, in particular $i_0(f)$ is zero on $\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})\times\{0\}$. Then $\forall x \in \mathcal{G}^{(0)}$, $\pi^{\mathcal{G}^T}_{(x,0)}(i_0(f)) = 0$. Using this fact, and the fact that $\|\pi^{\mathcal{G}^T}_{(x,t)}(f)\| = \|\pi^{\mathcal{G}\times]0,1]}_{(x,t)}(f)\|$ for $x\in\mathcal{G}^{(0)}$ and $t\neq 0$, we get:

$$\begin{aligned}\|i_0^r(f)\|_{r,\mathcal{G}^T} &= \sup_{(x,t)\in\mathcal{G}^{(0)}\times[0,1]} \|\pi^{\mathcal{G}^T}_{(x,t)}(f)\| = \sup_{(x,t)\in\mathcal{G}^{(0)}\times]0,1]} \|\pi^{\mathcal{G}^T}_{(x,t)}(f)\| \\ &= \sup_{(x,t)\in\mathcal{G}^{(0)}\times[0,1]} \|\pi^{\mathcal{G}\times]0,1]}_{(x,t)}(f)\| = \|f\|_{r,\mathcal{G}\times]0,1]}.\end{aligned}$$

Then the map is continuous.

With the last propositions, the main point to show continuity were the vanishing of some representations. Then the supremum of norms over $\mathcal{G}\times[0,1]$ reduced to the supremum of norms over a subset. Now $\|.\|_1$ involve a supremum of integrals over the target fibers, we saw that fibers were preserved, then the vanishing argument is true with these integrals. Then we can do exactly the same proofs to conclude:

Property 3.1.7

The maps:

$$\begin{array}{ccc} \left(\mathcal{C}_c\left(\mathcal{G}^T\right), \|.\|_{1,\mathcal{G}^T}\right) & \longrightarrow & \left(C^*(\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)}), \|.\|_{1,\mathcal{N}}\right) \\ f & \mapsto & f_{|\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})} \\ \left(\mathcal{C}_c\left(\mathcal{G}^T\right), \|.\|_{1,\mathcal{G}^T}\right) & \longrightarrow & \left(C^*(\mathcal{G}), \|.\|_{1,\mathcal{G}}\right) \\ f & \mapsto & f_{|\mathcal{G}} \\ \left(\mathcal{C}_c\left(\mathcal{G}\times]0,1]\right), \|.\|_{1,\mathcal{G}\times]0,1]}\right) & \longrightarrow & \left(C^*(\mathcal{G}^T), \|.\|_{1,\mathcal{G}^T}\right) \\ f & \mapsto & f \end{array}$$

are continuous, and then they extend as:

$$ev_0^* : C^*(\mathcal{G}^T) \to C^*(\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})) \quad ; \quad ev_1^* : C^*(\mathcal{G}^T) \to C^*(\mathcal{G}) \quad ; \quad i_0^* : C^*(\mathcal{G}\times]0,1]) \to C^*(\mathcal{G}^T)$$

Until now we did the same thing with the full algebras and the reduced ones. And in practice they are equal really often, this is the purpose of the amenability notion. This is a notion we will not study.
Now we focus on full algebras, and we build a link between them:

Property 3.1.8

The full algebras previously defined and their maps fit into a short exact sequence :

$$0 \longrightarrow C^*(\mathcal{G}\times]0,1]) \xrightarrow{i_0^*} C^*(\mathcal{G}^T) \xrightarrow{ev_0^*} C^*(\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})) \longrightarrow 0$$

Here is a partial proof of this fact.

Proof

i_0 injective :

When we extended i_0 we saw that : $\forall f \in \mathcal{C}_c\left(\mathcal{G}\times]0,1]\right), \|i_0(f)\|_{r,\mathcal{G}^T} = \|f\|_{r,\mathcal{G}\times]0,1]}$.
Then if $f \in C_r^*(\mathcal{G}\times]0,1])$ there exist $(f_n)_n$ a sequence of $\mathcal{C}_c\left(\mathcal{G}\times]0,1]\right)$ such that

$f_n \xrightarrow{\|\cdot\|_{r,\mathcal{G}\times]0,1]}} f$. But i_0 is continuous, then $i_0(f_n) \xrightarrow{\|\cdot\|_{r,\mathcal{G}^T}} i_0(f)$. Then:

$$\|f\|_{r,\mathcal{G}\times]0,1]} = \lim_{n\to\infty} \|f_n\|_{r,\mathcal{G}\times]0,1]} = \lim_{n\to\infty} \|i_0(f_n)\|_{r,\mathcal{G}^T} = \|i_0(f)\|_{r,\mathcal{G}^T}.$$

Then i_0 is an isometry between the reduced C^*-algebras, then is injective.

e_0 surjective :

Using the appendix theorem, we have a tubular neighbourhood of $\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)}) \times \{0\}$ in $\mathcal{G}^T$.
Then let $U \subset \mathcal{G}^T$ open set of compact closure containing $Supp(f)$, let $p : E \longrightarrow Supp(f)$ vector bundle and Φ a tubular adapted diffeomorphism :

$$\begin{array}{cccc} \Phi : & U \subset \mathcal{G}^T & \stackrel{\cong}{\longrightarrow} & V \subset E \\ & (x,v,0) & \mapsto & ((x,v,0),0) \\ & (g,t) & \mapsto & ((\varphi(g,t),0),\psi(g,t)) \end{array}$$

, with $\varphi = pr_1 \circ p \circ \Phi$ continuous map with its values in $\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})$.
In particular, if (g,t) goes to $(x,v,0)$ in $U \subset \mathcal{G}^T$, then $(\varphi(g,t),0)$ goes to $(x,v,0)$ in E.

Then let $f \in \mathcal{C}_c\left(\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)}\right)$. Let χ a plateau function with :

$$0 \leqslant \chi \leqslant 1 \quad ; \quad \chi_{|\mathcal{G}^T\setminus U} = 0 \quad ; \quad \chi_{|Supp(f)} = 1.$$

in particular, χ has compact support. Then let

$$\begin{array}{cccc} \widetilde{f} : & \mathcal{G}^T & \longrightarrow & \mathbb{C} \\ & (x,v,0) & \mapsto & f(x,v) \\ & (g,t) & \mapsto & f(\varphi(g,t))\chi(g,t) \end{array}$$

, $Supp(\widetilde{f}) \subset Supp(\chi)$, then $\widetilde{f}$ has compact support. Moreover $\widetilde{f}$ is continuous, then $\widetilde{f} \in \mathcal{C}_c\left(\mathcal{G}^T\right)$.
By contruction $ev_0(\widetilde{f}) = f$.
Then the continuous map $ev_0 : \mathcal{C}_c\left(\mathcal{G}^T\right) \longrightarrow \mathcal{C}_c\left(\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})\right)$ is surjective.

We should extend this surjectivity to the map ev_0^* between the full $C^* - algebras$. To do this we need a better understanding of these algebras. This work will be done in PhD during the next months.

$Im\, i_0 \subset Ker\, ev_0$:

If $f \in Im\, i_0$, then f vanishes on $\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)}) \times \{0\}$ and then $f \in Ker\, e_0$.

$Ker\, ev_0 \subset Im\, i_0$:

Admitted

Now using the contractibility of the intervam $]0,1]$, we can deduce that:

Property 3.1.9 Admitted

The C^*-algebra $C^*_r(\mathcal{G}\times]0,1])$ is contractible.
In particular it has trivial K-groups.

Then we can use the six terms exact sequence theorem in K-theory which provide the following exact sequence:

$$\begin{array}{ccccc} K_0(C^*(\mathcal{G}\times]0,1])) & \longrightarrow & K_0(C^*(\mathcal{G}^T)) & \xrightarrow{K_0(ev_0^*)} & K_0(C^*(\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)}))) \\ \uparrow & & & & \downarrow \\ K_1(C^*(\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)}))) & \longleftarrow & K_1(C^*(\mathcal{G}^T)) & \longleftarrow & K_1(C^*(\mathcal{G}\times]0,1])) \end{array}$$

But because $K_1(C^*(\mathcal{G}\times]0,1])) = K_0(C^*(\mathcal{G}\times]0,1])) = 0$, the exactness provides us the fact that:

Property 3.1.10

The map $K_0(ev_0^*) : K_0(C^*(\mathcal{G}^T)) \longrightarrow K_0(C^*(\mathcal{N}(\mathcal{G},\mathcal{G}^{(0)})))$ is an isomorphism.

3.2 Connes's tangent groupoïd

Definition 3.2.1 Connes's tangent groupoïd

Let M a smooth manifold, we call *Connes's tangent groupoïd* the tangent groupoïd associated to the pair Lie groupoïd $M \times M \rightrightarrows M$ after use of the identification $\mathcal{N}(M\times M, \Delta_M) \cong TM$.
We denote it G_M^{tan} and it has the form $(TM \times \{0\} \sqcup M \times M\times]0,1]) \rightrightarrows (M \times [0,1])$.

Using the same process as in the *Deformation to the Normal Cone* section, we can build charts for this groupoïd.

To simplify the notations and the process, we will suppose that M admit a global chart (M,φ).
Let $n = dimM$.
We define the map $\begin{array}{cccc} \tilde{\varphi}: & M\times M & \to & \mathbb{R}^{2n} \\ & (x,y) & \mapsto & \left(\frac{\varphi(x)+\varphi(y)}{2}, \frac{\varphi(x)-\varphi(y)}{2}\right) \end{array}$, then $(M\times M, \tilde{\varphi})$ is a chart of $M \times M$ adatped to Δ_M.
Its differential is easily computable according to the differential of φ. Then we can pass to the quotient to get the normal differential:

$$\begin{array}{cccc} d_{\mathcal{N}}\tilde{\varphi}: & \mathcal{N}(M\times M,\Delta_M) & \to & \mathcal{N}(\mathbb{R}^{2n}, \mathbb{R}^n\times\{0\}) \\ & ((\xi,\xi),(v,w) \mod T_{(\xi,\xi)}\Delta_M) & \mapsto & \left((\varphi(\xi),0), \left(\frac{d_\xi\varphi(v)+d_\xi\varphi(w)}{2}, \frac{d_\xi\varphi(v)-d_\xi\varphi(w)}{2}\right)\right) \end{array}.$$

Then we use the identifications $\beta_M : \mathcal{N}(M\times M,\Delta_M) \cong TM$ and $\alpha_{\mathbb{R}^n} : \mathcal{N}(\mathbb{R}^{2n},\mathbb{R}^n\times\{0\}) \cong T\mathbb{R}^n$ defined previously to get a map from TM to $T\mathbb{R}^n$ that we will also denote $d_{\mathcal{N}}\tilde{\varphi}$ for more simplicity.
Then we get:

$$\begin{array}{cccc} d_{\mathcal{N}}\tilde{\varphi}: & TM & \longrightarrow & T\mathbb{R}^n = \mathbb{R}^{2n} \\ & (\xi,v) & \mapsto & (\varphi(\xi), \frac{1}{2}d_\xi\varphi(v)) \end{array}$$

Then the maps $\Psi^{-1} \circ \mathscr{D}(\tilde{\varphi})$ gives us a chart of $(TM \times \{0\} \sqcup M \times M \times \mathbb{R}^*)$

$$(\xi, v, 0) \xmapsto{\mathscr{D}(\tilde{\varphi})} (\varphi(\xi), \tfrac{1}{2} d_\xi \varphi(v), 0) \xmapsto{\Psi^{-1}} (\varphi(\xi), \tfrac{1}{2} d_\xi \varphi(v), 0)$$

$$(x, y, t) \longmapsto \left(\tfrac{\varphi(x)+\varphi(y)}{2}, \tfrac{\varphi(x)-\varphi(y)}{2}, t \right) \longmapsto \left(\tfrac{\varphi(x)+\varphi(y)}{2}, \tfrac{1}{2} \tfrac{\varphi(x)-\varphi(y)}{t}, t \right)$$

that we could restrict to the Connes's tangent groupoïd. We can also invert it to get a parametrization, then we get:

Property 3.2.2

From a differential manifold M with a chart (M, φ) we can define a chart, or a parametrization of the Connes's tangent groupoïd G_M^{tan} as follows:

Chart:

G_M^{tan}	$\to$	$\mathbb{R}^n \times \mathbb{R}^n \times [0,1]$
$(\xi, v, 0)$	$\mapsto$	$(\varphi(\xi), \frac{1}{2} d_\xi \varphi(v), 0)$
(x, y, t)	$\mapsto$	$\left(\frac{1}{2}(\varphi(x) + \varphi(y)), \frac{1}{2} \frac{\varphi(x)-\varphi(y)}{t}, t \right)$

Parametrization:

$\mathbb{R}^n \times \mathbb{R}^n \times [0,1]$	$\to$	G_M^{tan}
$(\xi, v, 0)$	$\mapsto$	$(\varphi^{-1}(\xi), 2(d_{\varphi^{-1}(\xi)}\varphi)^{-1}(v), 0)$
(x, y, t)	$\mapsto$	$(\varphi^{-1}(x + ty), \varphi^{-1}(x - ty), t)$

Using that chart, we can understand how the manifold $M \times M$ deform itself to TM:

Property 3.2.3

Let $(x_n)_n, (y_n)_n$ sequences in a differential manifold M, and $(t_n)_n$ sequence of $\mathbb{R}^{*+}$ and let $(\xi, v) \in TM$.
The convergence $(x_n, y_n, t_n) \underset{n\to+\infty}{\longrightarrow} (\xi, v, 0)$ in G_M^{tan} is equivalent to the convergences

$$\begin{cases} t_n \underset{n\to+\infty}{\longrightarrow} 0 \\ x_n,\ y_n \underset{n\to+\infty}{\longrightarrow} \xi \text{ in } M \\ \frac{\varphi(x_n)-\varphi(y_n)}{t_n} \underset{n\to+\infty}{\longrightarrow} d_\xi \varphi(v) \end{cases}.$$

Proof

If $(x_n, y_n, t_n) \underset{n\to+\infty}{\longrightarrow} (\xi, v, 0)$ in G_M^{tan}, then reading this convergence in the charts we have $\frac{\varphi(x_n)+\varphi(y_n)}{2} \underset{n\to+\infty}{\longrightarrow} \varphi(\xi)$, $\frac{\varphi(x_n)-\varphi(y_n)}{t_n} \underset{n\to+\infty}{\longrightarrow} d_\xi\varphi(v)$ and $t_n \underset{n\to+\infty}{\longrightarrow} 0$.
Because $\frac{\varphi(x_n)-\varphi(y_n)}{t_n}$ has a limit while t_n goes to 0, then $\varphi(x_n) - \varphi(y_n) \underset{n\to+\infty}{\longrightarrow} 0$.
The sum $\varphi(x_n) + \varphi(y_n)$ also has a limit, then adding and substracting them we get that $\varphi(x_n)$ and $\varphi(y_n)$ have limits too. The same limit because the difference goes to 0. Then $\varphi(x_n), \varphi(y_n) \underset{n\to+\infty}{\longrightarrow} \varphi(\xi)$. Which means that x_n and y_n goes to ξ in M.

The converse is immediate.

Let's use what we did in the previous section on the Connes's tangent groupoïd. We have these maps:

$$K_0(C^*(TM)) \xrightarrow{K_0(ev_0^*)^{-1}} K_0(C^*(G_M^{tan})) \xrightarrow{K_0(ev_1^*)} K_0(C^*(M \times M)) \ .$$

To finally get the result we wanted, we will admit the following property:

Property 3.2.4 Amenability - Admitted

For the groupoïds $TM \rightrightarrows M$ and $M \times M \rightrightarrows M$ there are canonical isomorphisms:

$$C_r^*(TM) \cong C^*(TM) \text{ and } C_r^*(M \times M) \cong C^*(M \times M).$$

Then everything we proved for an algebra is true for the other one in this case.

Then we can use the examples of completion that we saw before to state that $C_r^*(TM) \cong C_0(T^*M)$ (with the isomorphism $\widetilde{\mathcal{F}}$) and $C_r^*(M \times M) \cong \mathcal{K}(L^2(M))$ (with $\widetilde{\pi_{x_0}}$).

Then knowing that, by definition $K^0_{top}(T^*M) = K_0(C_0(T^*M))$.
Knowing also that the Hilbert space $L^2(M)$ is the inductive limit of the spaces $M_n(\mathbb{C})$ and that $K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$, we can state that $K_0(\mathcal{K}(L^2(M))) \cong \mathbb{Z}$.

Then putting everything on a diagram, we finally get

$$\begin{array}{c} K^0_{top}(T^*M) = K_0(C_0(T^*M)) \xrightarrow{[K_0(\widetilde{\mathcal{F}}) \circ K_0(ev_0)]^{-1}} K_0(C^*(G_M^{tan})) \xrightarrow{K_0(e_1)} K_0(C^*(M \times M)) \xrightarrow{K_0(\widetilde{\pi_{x_0}})} K_0(\mathcal{K}(L^2(M))) \\ \hfill \downarrow \cong \\ \hfill \mathbb{Z} \end{array}$$

Chapter 4

Appendix

Theorem 4.0.1 Extension of continuous linear map

Let E and F two normed vector spaces. We suppose F to be complete. Let $A \subset E$ a dense sub vector space and $f : A \to F$ a continuous linear map.
Then there is a unique continuous linear extension $\tilde{f} : E \to F$, moreover $\|f\| = \|\tilde{f}\|$.

Proof

Unicity:

Let g_1 and g_2 two continuous extensions of f. We set the continuous map:
$$g_1 \times g_2 : \begin{array}{ccc} E & \longrightarrow & F \times F \\ x & \mapsto & (g_1(x), g_2(x)). \end{array}$$
Then $\{g_1 = g_2\}$ is closed, and contains A. Then it contains $\overline{A} = E$. Then $g_1 = g_2$.

Existence:

Let $x \in E$. Then we have $(x_n)_n$ a sequence of A such that $x_n \underset{n\to+\infty}{\longrightarrow} x$.
Then $\|f(x_p) - f(x_q)\| \leqslant \|f\|\|x_p - x_q\| \longrightarrow 0$. Then $(f(x_n))_n$ is a Cauchy sequence of F. Then a convergent one. We call its limit $\tilde{f}(x)$.
Now our purpose is to define a continuous linear map $\tilde{f}$ using this process. Let's check that it is well defined this way. Let $(y_n)_n$ another sequence which converges to x. Then:

$$\begin{aligned} \|f(y_n) - \tilde{f}(x)\| &\leqslant \|f(y_n) - f(x)\| + \|f(x) - f(x_n)\| + \|f(x_n) - \tilde{f}(x)\| \\ &\leqslant \|f\|(\|y_n - x\| + \|x - x_n\|) + \|f(x_n) - \tilde{f}(x)\| \underset{n\to+\infty}{\longrightarrow} 0. \end{aligned}$$

Then $f(y_n) \underset{n\to+\infty}{\longrightarrow} \tilde{f}(x)$. Then $\tilde{f}$ is well defined.
Morevoer, for $x \in A$, we can choose the constant sequence equal to x, then $\tilde{f}$ is an extension of f.

Using the same sequence argument to approximate $\tilde{f}$ by f, we show that $\tilde{f}$ is linear, continuous, and $\|f\| = \|\tilde{f}\|$.

Definition 4.0.2 Tubular Neighbourhood

Let $(X, Y) \in \mathcal{C}_2^\infty$. An open neighbourhood U of Y in X is said to be tubular if there exists a vector bundle $p : E \to Y$, a neighbourhood V of Y (of its zero section actually) in E and a diffeomorphism $\Phi : U \overset{\cong}{\to} V \subset E$ such that $\Phi(x) = (x, 0)$ if and only if $x \in Y$.

Theorem 4.0.3 Tubular Neighbourhood theorem - admitted

Let $(X, Y) \in \mathcal{C}_2^\infty$. The submanifold Y admit a tubular neighbourhood.

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